Journal of Mathematical Economics 92 (2021) 10–21

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Optimally biased Tullock contests[☆]

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ARTICLE INFO

Article history:
Received 26 March 2020
Received in revised form 14 October 2020
Accepted 15 October 2020
Available online 28 October 2020

Keywords:
Contest design
Tullock contest
Research tournament
Optimal bias

ABSTRACT

This paper examines optimally biased Tullock contests. We consider a multi-player Tullock contest in which players differ in their prize valuations. The designer is allowed to impose identity-dependent treatments – i.e., multiplicative biases – to vary their relative competitiveness. The literature has been limited, because a closed-form solution to the equilibrium is in general unavailable when the number of contestants exceeds two, which nullifies the usual implicit programming approach. We develop an algorithmic technique adapted from the general approach of Fu and Wu (2020) and obtain a closed-form solution to the optimum that addresses a broad array of design objectives. We further analyze a resource allocation problem in a research tournament and adapt Fu and Wu's (2020) approach to this noncanonical setting. Our analysis paves the way for future research in this vein.

1. Introduction

In a contest, contenders sink irreversible effort or costly bids to vie for limited prizes, while their competitive outlays are nonrefundable regardless of the outcome. Such competitions can be exemplified by a plethora of examples, ranging from electoral competitions, lobbying, R&D races, and college admissions to sporting events. A voluminous economics literature has been developed to investigate contestants' strategic behavior and the optimal design of contests for various goals.

This paper studies optimal contest design with contestants who differ in strength. Such heterogeneity affords the designer the flexibility to administer identity-dependent treatments that manipulate their relative competitiveness and bias the competition. Consider, for instance, the widespread practice of affirmative action in college admissions. Similarly, incumbent workers in firms are often ex ante preferred to external candidates when they compete for a vacancy.

We focus on the popularly adopted Tullock contest model and develop an algorithmic technique – which extends the general approach proposed by Fu and Wu (2020) – to solve for the optimally biased contest in closed form that addresses a wide variety of concerns. Imagine an asymmetric multi-player winner-take-all contest in which $n \geq 2$ contestants differ in their prize valuations. With effort entries $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, contestant i wins with a probability

$$p_i(\boldsymbol{x}, \boldsymbol{\alpha}) := \frac{\alpha_i f(x_i)}{\sum_{j=1}^{n} \alpha_j f(x_j)},$$

with $\alpha_i \geq 0$ for all $i \in \{1, \ldots, n\}$; the function $f(\cdot)$ is typically labeled the *impact function* of the contest, and takes the form of $f(x) = x^r$, with $r \in (0, 1]$ in Tullock settings. The set of weights $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$ is a design variable or contest rule to be chosen by the designer prior to the competition.

The literature on optimally biased contests has typically focused on two-player settings and/or restricted design objectives. The conventional optimization approach requires an equilibrium solution of the contest under every possible set of biases $\boldsymbol{\alpha}$. However, a solution is in general unavailable when three or more players are involved, except for the case of lottery contests, i.e., $r = 1$. Based on the equilibrium characterization of Stein (2002), Franke et al. (2013) make a pioneering contribution to solving for optimal biases that maximize total effort in a multi-player lottery contest that allows for $n \geq 3$. Fu and Wu (2020) introduce an alternative avenue for the optimization problem: They let the designer choose the equilibrium winning probability

☆ We are grateful to Editor-in-Chief Andrés Carvajal, Associate Editor Caroline Thomas, and three anonymous reviewers for constructive and insightful comments. We thank Jörg Franke, Wolfgang Leininger, Jianpei Li, Tore Nilssen, Paul Schweinzer, and seminar/conference participants at the University of International Business and Economics, the 4th Annual Conference on Contests: Theory and Evidence, and the 2018 International Conference on Economic Theory and Applications (Chengdu) for helpful discussions, suggestions, and comments. We thank Yuxuan Zhu for excellent research assistance. Wu thanks the National Natural Science Foundation of China (No. 71803003) and the seed fund of the School of Economics, Peking University, for financial support. Any errors are our own.

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https://doi.org/10.1016/j.jmateco.2020.10.004
0304-4068/

distribution as the design variable to maximize a more general objective function – with total effort maximization as a special case – and then show that every equilibrium winning probability distribution can be induced by a set of weights $\boldsymbol{\alpha}$. This allows them to characterize the optimum without solving for the equilibrium.

Our paper extends Fu and Wu (2020) in two dimensions. First, they consider a general concave impact function $f(\cdot)$, which limits their attention to the qualitative properties of the optimal contest. In contrast, we examine in depth a Tullock setting and develop an algorithm to obtain a closed-form solution to the optimal biases $\boldsymbol{\alpha}$ and the set of active contestants – i.e., those who expend strictly positive efforts – in the optimum. This enables handy comparative statics of the optimum with respect to environmental factors. Second, we analyze a noncanonical setting for contest design that departs from the framework of Fu and Wu (2020), thereby expanding the scope of application for their approach.

We first construct a contest design problem in which the designer cares not only about total effort – which is commonly assumed in the contest literature – but also the selection efficiency and/or "closeness" of the competition.[1] Formally, she maximizes a convex combination of aggregate effort, the expected ability of the winner, and the variance of contestants' equilibrium winning probability distribution. We detail an algorithm for the analysis, which paves the way for future studies of optimally biased Tullock contests.

We then analyze an optimal resource allocation problem in a research tournament à la Fullerton and McAfee (1999), which is strategically equivalent to a Tullock contest (see Baye and Hoppe, 2003; Fu and Lu, 2012). We allow a sponsor to split a fixed amount of productive resources – e.g., research funding – among firms to maximize the expected quality of the winning product. Lichtenberg (1990), for instance, documents the fact that extensive subsidies are provided by the United States Department of Defense (DoD) to assist private military technology firms that compete for defense procurement contracts.[2] The subsidy exemplifies a "technology-based" (Kirkegaard, 2020) preferential treatment, as it not only varies firms' relative competitiveness, but also affects the recipient's actual productivity. This optimization problem differs from the setting delineated by Fu and Wu (2020), because the design variable – i.e., the resource allocation profile – influences actual output and thus directly enters the designer's objective function.

The remainder of the paper is organized as follows: In Section 2, we describe the baseline model, lay out the optimal contest design problem, solve for the optimal contest, and present comparative statics. In Section 3, we extend the model to consider optimal resource allocation in R&D contests. Section 4 concludes the paper.

2. Optimal design of biased Tullock contests

In this section, we first set up the contest model and the optimal design problem. We then carry out the optimal analysis.

[1] In a corporate succession race, the board cares not only about candidates' productive efforts but also the winning candidate's quality. Similarly, in a sporting event, the organizer benefits not only from athletes' effort but also from suspense regarding the outcome (Chan et al., 2008).

[2] In the famous joint strike fighter (JSF) competition, the DoD financially sponsored Lockheed Martin's and Boeing's prototype development.

2.1. The model

There are $n \geq 2$ risk-neutral contestants competing for a prize. The prize bears a value $v_i > 0$ for each contestant $i \in \mathcal{N} \equiv \{1, \ldots, n\}$, with $v_1 \geq \cdots \geq v_n > 0$, which is commonly known. To win the prize, contestants simultaneously submit their effort entry $x_i \geq 0$. One's bid incurs a unity marginal effort cost. It should be noted that modeling contestants' heterogeneity through different prize valuations is equivalent to assuming different marginal effort costs.[3]

We consider a standard Tullock contest: For a given effort profile $\boldsymbol{x} \equiv (x_{1,} \ldots, x_n)$, a contestant i wins the prize with a probability

$$p_i(\boldsymbol{x}, \boldsymbol{\alpha}) := \begin{cases} \frac{\alpha_i x_i^r}{\sum_{j=1}^n \alpha_j x_j^r} & \text{if } \sum_{j=1}^n \alpha_j x_j^r > 0, \\ \frac{\alpha_i}{\sum_{j=1}^n \alpha_j} & \text{if } \sum_{j=1}^n \alpha_j x_j^r = 0, \end{cases} \quad (1)$$

with $\alpha_i \geq 0$ and $r \in (0, 1]$. The parameter $r \in (0, 1]$ measures the precision of the winner-selection mechanism. The set of weights $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$ is set by a contest designer as the contest rule prior to the competition.

Given the above contest success function (1) and the effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, contestant i's expected payoff is

$$\pi_i(\boldsymbol{x}, \boldsymbol{\alpha}) := p_i(\boldsymbol{x}, \boldsymbol{\alpha}) \cdot v_i - x_i.$$

2.2. Contest design: Mathematical Programming with Equilibrium Constraints (MPEC)

Prior to the contest, the designer, anticipating contestants' equilibrium plays, chooses contest rule $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n \setminus \{(0, \ldots, 0)\}$. Szidarovszky and Okuguchi (1997) and Cornes and Hartley (2005) establish the existence and uniqueness of a pure-strategy equilibrium in n-player Tullock contest games with $r \in (0, 1]$. The optimization problem thus yields a typical mathematical program with equilibrium constraints (MPEC): Contestants' equilibrium effort profile, $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, and the equilibrium winning probability distribution, $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$, are endogenously determined in the equilibrium as functions of $\boldsymbol{\alpha}$, which is set by the designer to maximize the following objective function:

$$\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) := \sum_{i=1}^n x_i + \psi \sum_{i=1}^n p_i v_i - \gamma \sum_{i=1}^n \left(p_i - \frac{\sum_{j=1}^n p_j}{n}\right)^2, \quad \text{with } \psi, \gamma \geq 0. \quad (2)$$

In the case of $\psi = \gamma = 0$, the above expression boils down to $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) = \sum_{i=1}^n x_i$, the objective of total effort maximization examined in the majority of studies on contest design. In addition to effort supply, the contest designer is allowed to care about the selection efficiency and/or closeness of the competition. The term $\sum_{i=1}^n p_i v_i$ strictly increases when a contestant with a higher prize valuation wins with a larger probability, which accommodates the concern about selection efficiency with $\psi > 0$.[4] Further, the term $\sum_{i=1}^n \left[p_i - (\sum_{j=1}^n p_j)/n\right]^2$ is the variance of contestants' equilibrium winning probabilities. With $\gamma > 0$, the designer prefers a less predictable competition, i.e., more suspense about the eventual outcome.[5]

[3] See the online appendix of Fu and Wu (2020) for a detailed analysis that demonstrates the isomorphism.

[4] See Meyer (1991), Hvide and Kristiansen (2003), Ryvkin and Ortmann (2008), and Fang and Noe (2019) for contest design for selection efficiency.

[5] See Chan et al. (2008) and Ely et al. (2015) for economics studies of suspense in competition. In addition, Fort and Quirk (1995), Szymanski (2003), and Runkel (2006) assume this objective in two-player contest settings.

2.3. Analysis

The traditional implicit programming approach has no bite in our context. It requires a closed-form equilibrium solution for every possible contest rule $\boldsymbol{\alpha}$ in order to rewrite the design objective as a function of $\boldsymbol{\alpha}$. However, a multi-player asymmetric Tullock contest game is, in general, unsolvable.

The alternative approach of Fu and Wu (2020), which allows us to bypass this technical difficulty, can be summarized as follows. It first establishes a correspondence between contestants' effort profile and winning probability distribution in the unique equilibrium of each contest game. Based on the correspondence, the design objective can be rewritten as a function of the equilibrium winning probability distribution only. Next, the designer assigns equilibrium winning probabilities to contestants to maximize the reformulated objective function. Finally, they show that a contest rule can be identified to induce the desirable equilibrium winning probability distribution. This approach yields qualitative implications for optimal contests in the general framework of Fu and Wu (2020), but it does not provide straightforward instruction for the analysis in more structured settings. Therefore, in this paper, we develop a five-step algorithm based on Fu and Wu (2020) that leads to a closed-form solution to the optimum in Tullock contests.

2.3.1. Reformulation of the designer's problem

With a contest success function (1), the first-order condition $\partial\pi_i(\boldsymbol{x},\boldsymbol{\alpha})/\partial x_i = 0$ for an *active* contestant i – who exerts a strictly positive amount of effort – is

$$\frac{\sum_{j\neq i}\alpha_j(x_j)^r}{\left[\sum_{j=1}^n \alpha_j(x_j)^r\right]^2}\times r(x_i)^{r-1} = \frac{1}{\alpha_i v_i}.$$

The above equilibrium condition, together with the winning probability $p_i(\boldsymbol{x},\boldsymbol{\alpha})$ specified in Eq. (1), implies immediately a fundamental correspondence[6]

$$x_i = p_i(1-p_i)v_i r. \quad (3)$$

From Eq. (3), an equilibrium effort profile $\boldsymbol{x}$ is uniquely associated with a distribution of contestants' equilibrium winning probabilities $\boldsymbol{p}$. The contest objective (2) can accordingly be rewritten as

$$\begin{aligned}\Lambda\big(\boldsymbol{x}(\boldsymbol{p},\boldsymbol{v}),\boldsymbol{p},\boldsymbol{v}\big) :&= \sum_{i=1}^n [rp_i(1-p_i)v_i] + \psi\sum_{i=1}^n p_i v_i \\ &\quad - \gamma\sum_{i=1}^n\left(p_i - \frac{\sum_{j=1}^n p_j}{n}\right)^2 \\ &= \sum_{i=1}^n\left[p_i\left(1+\frac{\psi}{r}-p_i\right)v_i r\right] \\ &\quad - \gamma\sum_{i=1}^n (p_i)^2 + \frac{\gamma}{n}, \text{ with } \psi,\gamma\geq 0. \end{aligned} \quad (4)$$

The optimization problem is readily reformulated. We treat the distribution of winning probabilities $\boldsymbol{p}$ as the design variable: The designer chooses an equilibrium winning probability distribution, $\boldsymbol{p}\equiv(p_1,\ldots,p_n)$, to maximize the objective function (4), subject to the constraints:

$$\sum_{i=1}^n p_i = 1, \text{ and } p_i \geq 0, \text{ for all } i\in\mathcal{N}. \quad (5)$$

With the reformulated objective function (4), the optimization problem boils down to a constrained quadratic programming problem. A maximizer automatically exists given that the choice set, defined by (5), is an $(n-1)$-dimensional simplex.

2.3.2. Optimization

Denote by $\boldsymbol{p}^*\equiv(p_1^*,\ldots,p_n^*)$ the optimal winning probabilities that maximize contest objective (4). The following useful property can be established.

Lemma 1 (*Monotone Winning Odds Ranking*)**.** *Suppose that $v_i\geq v_j$; then $p_i^*\geq p_j^*$.*

It is noteworthy that Lemma 1 differs from Theorem 4 and Proposition 3 of Fu and Wu (2020): The former does not provide a ranking of active contestants' winning odds in the optimum, while the latter restricts its attention to the case of total effort maximization. By Lemma 1, an ex ante stronger contestant – i.e., a player with a larger v_i – must win with a (weakly) higher probability in the optimal contest. This in turn implies that whenever the designer aims to exclude contestants – i.e., by assigning zero or excessively small weights to discourage participation – she must target the ex ante weakest.

The fact that contestants may choose to stay inactive – i.e., exerting zero effort – under certain $\boldsymbol{\alpha}$, leads to a nonsmooth optimization problem in contest design. As a result, setting $\boldsymbol{\alpha}$ involves a hidden problem of optimally selecting participants, which has plagued the equilibrium analysis of contest games. Lemma 1 inspires an algorithm that allows us to identify the set of active contestants in the optimum and then solve for the optimal contest rule. A sketch analysis is laid out below – which proceeds in five steps – and more details are provided in the Appendix.

1. We construct a sequence of auxiliary problems $(\mathcal{P}_m)$: For each $m=1,\ldots,n$, the contest designer maximizes objective (4) subject to $\sum_{i=1}^n p_i = 1$, ignoring the nonnegativity constraint $p_i\geq 0$ for $i\in\{1,\ldots,m\}$ and setting $p_i=0$ for $i\in\mathcal{N}\setminus\{1,\ldots,m\}$.
2. The solution to the auxiliary optimization problem $(\mathcal{P}_m)$, which we denote by $\check{\boldsymbol{p}}^m\equiv(\check{p}_1^m,\ldots,\check{p}_n^m)$, can be obtained explicitly by computing the first-order conditions, and is given by

$$\check{p}_i^m = \begin{cases} \dfrac{\psi+r}{2r}\left(\dfrac{v_i r}{v_i r+\gamma} - \dfrac{1}{v_i r+\gamma} \times \dfrac{\sum_{j=1}^m \frac{v_j r}{v_j r+\gamma} - \frac{2r}{\psi+r}}{\sum_{j=1}^m \frac{1}{v_j r+\gamma}}\right) & \text{for } i\in\{1,\ldots,m\}, \\ 0 & \text{for } i\in\mathcal{N}\setminus\{1,\ldots,m\}. \end{cases} \quad (6)$$

   Close inspection of the above solution (6) would reveal $\check{p}_1^m\geq\cdots\geq\check{p}_m^m$. Therefore, $\check{p}_i^m\in(0,1]$ for all $i\in\{1,\ldots,m\}$ if $\check{p}_m^m>0$, given the constraint of $\sum_{i=1}^n\check{p}_i^m = 1$. In other words, $\check{\boldsymbol{p}}^m$ with $\check{p}_m^m>0$ satisfies the nonnegativity constraints $p_i\geq 0$ for all $i\in\mathcal{N}$ imposed by the original optimization problem.
3. By Lemma 1, the optimal contest must exclude contestants from the bottom. Therefore, the solution to the original maximization problem – i.e., $\boldsymbol{p}^*\equiv(p_1^*,\ldots,p_n^*)$ – must be in the form of $(p_1,\ldots,p_\kappa,0,\ldots,0)$, with $p_i>0$ for $i\leq\kappa$, where $\kappa\in\{1,\ldots,n\}$ denotes the number of active contestants in the optimum and will be determined in the next step. The above analysis implies that the solution to the original maximization problem must coincide with that of the auxiliary problem $(\mathcal{P}_\kappa)$, i.e., $\boldsymbol{p}^*=\check{\boldsymbol{p}}^\kappa\equiv(\check{p}_1^\kappa,\ldots,\check{p}_\kappa^\kappa,0,\ldots,0)$.

[6] Note that the correspondence also holds for an *inactive* contestant who exerts zero effort: If a contestant stands zero chance of winning in a Tullock contest, he must exert zero effort.

4. Note that an auxiliary problem $(\mathcal{P}_{m+1})$ can be obtained by dropping the equality constraint $p_{m+1} = 0$ from the auxiliary problem $(\mathcal{P}_m)$. Therefore, $\Lambda\left(\boldsymbol{x}(\check{\boldsymbol{p}}^m, \boldsymbol{v}), \check{\boldsymbol{p}}^m, \boldsymbol{v}\right)$ is increasing in m. We can thus conclude that κ is the maximal $m \in \{1, \ldots, n\}$ such that $\check{p}_m^m > 0$, i.e.,

$$\kappa = \max\left\{m = 1, \ldots, n \,\middle|\, \check{p}_m^m > 0\right\} \equiv \max\left\{m = 1, \ldots, n \,\middle|\, \frac{\sum_{j=1}^m \frac{v_j r}{v_j r+\gamma} - \frac{2r}{\psi+r}}{\sum_{j=1}^m \frac{1}{v_j r+\gamma}} < v_m r\right\}. \tag{7}$$

5. Finally, any equilibrium winning probability distribution $\boldsymbol{p} \equiv (p_1, \ldots, p_n) \in \Delta^{n-1}$, with $p_1 \in (0, 1)$, can be induced in equilibrium by the following set of biases $\boldsymbol{\alpha}(\boldsymbol{p}) \equiv (\alpha_1(\boldsymbol{p}), \ldots, \alpha_n(\boldsymbol{p}))$:

$$\alpha_i(\boldsymbol{p}) = \begin{cases} \frac{(p_i)^{1-r}}{[(1-p_i)v_i]^r} \Big/ \left[\sum_{j\in\mathcal{N}_+(\boldsymbol{p})} \frac{(p_j)^{1-r}}{[(1-p_j)v_j]^r}\right] & \text{if} \quad p_i > 0, \\ 0 & \text{if} \quad p_i = 0, \end{cases} \tag{8}$$

where $\mathcal{N}_+(\boldsymbol{p}) := \{i = 1, \ldots, n \mid p_i > 0\}$ denotes the set of active contestants. In the extreme case in which $p_1 = 1$, we set $\alpha_1 = 1$ and $\alpha_i = 0$ for all $i \in \{2, \ldots, n\}$. The solution to the optimal biases, which we denote by $\boldsymbol{\alpha}^* \equiv (\alpha_1^*, \ldots, \alpha_n^*)$, follows accordingly.[7]

The following can then be obtained, which provides a solution for the optimal biases $\boldsymbol{\alpha}^*$, the number of active contestants in the optimum κ, and the corresponding equilibrium winning probabilities $\boldsymbol{p}^*$.

Theorem 1 (*Optimal Contest Concerning Total Effort, Selection Efficiency, and Closeness*)**.** *The equilibrium winning probabilities $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$ under the optimal contest are given by*

$$p_i^* = \begin{cases} \frac{\psi+r}{2r}\left(\frac{v_i r}{v_i r+\gamma} - \frac{1}{v_i r+\gamma} \times \frac{\sum_{j=1}^{\kappa} \frac{v_j r}{v_j r+\gamma} - \frac{2r}{\psi+r}}{\sum_{j=1}^{\kappa} \frac{1}{v_j r+\gamma}}\right) & \textit{for} \quad i \leq \kappa, \\ 0 & \textit{for} \quad i > \kappa, \end{cases} \tag{9}$$

where κ is given by Eq. (7). With $|\mathcal{N}_+(\boldsymbol{p}^*)| \geq 2$, the corresponding weights $\boldsymbol{\alpha}^* \equiv (\alpha_1^*, \ldots, \alpha_n^*)$ that induce $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$ are given by[8]

$$\alpha_i^* = \begin{cases} \frac{(p_i^*)^{1-r}}{[(1-p_i^*)v_i]^r} \Big/ \left[\sum_{j=1}^{\kappa} \frac{\left(p_j^*\right)^{1-r}}{\left[\left(1-p_j^*\right)v_j\right]^r}\right] & \textit{if} \quad p_i^* > 0, \\ 0 & \textit{if} \quad p_i^* = 0. \end{cases} \tag{10}$$

We now briefly interpret the result and analysis. As mentioned above, the contest design entails a nonsmooth optimization problem because contestants may choose to remain inactive in response to the prevailing contest rule. Lemma 1 indicates that the optimum keeps only bottom-ranked contestants inactive, which narrows down the search for the optimum and inspires the sequence of auxiliary problems $(\mathcal{P}_m)$. The solution for κ, the number of active contestants in the optimum, further leads us to obtain the equilibrium winning probability distribution $\boldsymbol{p}^*$ and, subsequently, the optimal biases $\boldsymbol{\alpha}^*$. We provide the following numerical example to illustrate the analysis.

[7] It should be noted that the weights $\boldsymbol{\alpha}(\boldsymbol{p})$ that induce each given $\boldsymbol{p}$ are not unique. Namely, the same equilibrium effort profile can be induced by scaling all $\alpha_i(\boldsymbol{p})$ up or down by a positive factor. Our construction normalizes the summation of the weights to unity, i.e., $\sum_{i=1}^n \alpha_i(\boldsymbol{p}) = 1$.

[8] In the trivial case of $|\mathcal{N}_+(\boldsymbol{p}^*)| = 1$, $\boldsymbol{p}^*$ can be induced by $\boldsymbol{\alpha}^* = (1, 0, \ldots, 0)$.

Table 1
Optimal contest under different combinations of (ψ, γ).

	p_1^*	p_2^*	p_3^*	p_4^*	p_5^*	p_6^*	κ
$\gamma = 0.2, \psi = 0.1$	0.2661	0.2377	0.2030	0.1597	0.1039	0.0296	6
$\gamma = 0.2, \psi = 0.2$	0.2791	0.2470	0.2078	0.1588	0.0957	0.0117	6
$\gamma = 0.2, \psi = 0.3$	0.2911	0.2552	0.2113	0.1565	0.0860	0	5
$\gamma = 0.1, \psi = 0.2$	0.2853	0.2522	0.2113	0.1595	0.0917	0	5
$\gamma = 0.2, \psi = 0.2$	0.2791	0.2470	0.2078	0.1588	0.0957	0.0117	6
$\gamma = 0.3, \psi = 0.2$	0.2734	0.2423	0.2047	0.1582	0.0992	0.0222	6
	α_1^*	α_2^*	α_3^*	α_4^*	α_5^*	α_6^*	κ
$\gamma = 0.2, \psi = 0.1$	0.1361	0.1456	0.1567	0.1698	0.1858	0.2059	6
$\gamma = 0.2, \psi = 0.2$	0.1386	0.1475	0.1577	0.1697	0.1842	0.2023	6
$\gamma = 0.2, \psi = 0.3$	0.1762	0.1864	0.1980	0.2116	0.2278	0	5
$\gamma = 0.1, \psi = 0.2$	0.1748	0.1856	0.1980	0.2123	0.2292	0	5
$\gamma = 0.2, \psi = 0.2$	0.1386	0.1475	0.1577	0.1697	0.1842	0.2023	6
$\gamma = 0.3, \psi = 0.2$	0.1375	0.1465	0.1571	0.1696	0.1849	0.2044	6

Example 1. Consider a contest with $n = 6$, $\boldsymbol{v} = (2, 1.8, 1.6, 1.4, 1.2, 1)$, and $r = 1$. We consider a case of $\gamma = 0.2$ and $\psi = 0.3$. Applying (6), we can calculate $\check{p}_m^m$ for each auxiliary problem $(\mathcal{P}_m)$, with $m \in \{1, \ldots, 6\}$. Note that $\check{p}_m^m > 0$ for $m = 1, \ldots, 5$ and $\check{p}_6^6 < 0$, which gives $\kappa = 5$ by (7). We can then obtain the equilibrium winning probability distribution associated with the optimum from the solution to the auxiliary problem $(\mathcal{P}_5)$, i.e., $p_i = \frac{\psi+r}{2r}\left\{\frac{v_i r}{v_i r+\gamma} - \frac{1}{v_i r+\gamma}\left[\sum_{j=1}^5\left(\frac{v_j r}{v_j r+\gamma} - \frac{2r}{\psi+r}\right)\right]\Big/\left(\sum_{j=1}^5 \frac{1}{v_j r+\gamma}\right)\right\}$ for $i = 1, \ldots, 5$ and $p_6 = 0$, which gives $\boldsymbol{p}^* = (0.2911, 0.2552, 0.2113, 0.1565, 0.0860, 0)$. Finally, we apply the rule of (10), which gives $\boldsymbol{\alpha}^* = (0.1762, 0.1864, 0.1980, 0.2116, 0.2278, 0)$.

The closed-form optimal solution enables comparative static analysis. A change in ψ or γ would vary the optimal bias rule $\boldsymbol{\alpha}^*$ and the associated equilibrium winning probability distribution $\boldsymbol{p}^*$. Further, a contestant's equilibrium winning probability p_i^* may drop to zero, which changes κ. Consider $\kappa := \kappa(\psi, \gamma)$ as a function of ψ, the concern for selection efficiency, and γ, that for closeness. The first corollary examines the comparative statics of equilibrium winning probabilities with respect to ψ and γ when κ remains unchanged, while the second concerns how the number of active contestants responds to changes in parameters.

Corollary 1 (*Comparative Statics of Equilibrium Winning Probabilities*)**.** *The following statements hold:*

i. Fix $\gamma \geq 0$ and suppose that $\psi_H > \psi_L \geq 0$, with $\kappa(\psi_H, \gamma) = \kappa(\psi_L, \gamma)$. There exists a threshold $\bar{v} \in [v_\kappa, v_1]$ such that $p_i^*(\psi_H, \gamma) \geq p_i^*(\psi_L, \gamma)$ if and only if $v_i \geq \bar{v}$.
ii. Fix $\psi \geq 0$ and suppose that $\gamma_H > \gamma_L \geq 0$, with $\kappa(\psi, \gamma_H) = \kappa(\psi, \gamma_L)$. There exists a threshold $\bar{\bar{v}} \in [v_\kappa, v_1]$ such that $p_i^*(\psi, \gamma_H) \geq p_i^*(\psi, \gamma_L)$ if and only if $v_i \leq \bar{\bar{v}}$.

By Corollary 1, when the designer is subject to a stronger concern for selection efficiency (resp. closeness) – i.e., a larger ψ (γ) – winning odds will be tilted in favor of stronger (weaker) contestants, and thus larger (smaller) weights tend to be assigned to them. We resort to the same numeric setting – i.e., $n = 6$, $\boldsymbol{v} = (2, 1.8, 1.6, 1.4, 1.2, 1)$ and $r = 1$ – as in Example 1 and vary the values of ψ and γ to illustrate the comparative statics. In Table 1, the upper panel lays out contestants' equilibrium winning probabilities $\boldsymbol{p}^*$ in the optimum, while the lower panel lists the corresponding biases $\boldsymbol{\alpha}^*$. Holding fixed $\gamma = 0.2$, when ψ increases from 0.1 to 0.2, κ remains at 6, but the weights for contestants 1–3 increase, which increase their equilibrium winning probabilities. In contrast, those for contestants 4–6 decrease, as shown in the first and second rows of both panels. The opposite pattern can be observed for an increasing γ, and we omit the discussion for brevity.

The second and third rows of the lower panel show that when ψ increases from 0.2 to 0.3, contestant 6 is entirely discouraged and κ reduces to 5: With greater concern for selection efficiency in place, the underdogs would be treated even less favorably and eventually excluded from the competition. This observation is formally established below.

Corollary 2 (*Comparative Statics of Number of Active Contestants*). *The following statements hold:*

i. *Fix $\gamma \geq 0$ and suppose that $\psi_H > \psi_L \geq 0$, then $\kappa(\psi_H, \gamma) \leq \kappa(\psi_L, \gamma)$.*
ii. *Fix $\psi \geq 0$ and suppose that $\gamma_H > \gamma_L \geq 0$, then $\kappa(\psi, \gamma_H) \geq \kappa(\psi, \gamma_L)$.*

Corollary 2 states that a stronger concern for selection efficiency (closeness) tends to lead the designer to exclude more (fewer) contestants. Taken together, the two comparative static results suggest that with greater concern for selection efficiency (closeness) in place, the contest designer is less (more) inclined to level the playing field. That is, she favors underdogs less (more) in terms of equilibrium winning probability.

3. Optimal resource allocation in R&D contests

We now consider a resource allocation problem in a research tournament à la Fullerton and McAfee (1999). As will be shown below, the model is strategically equivalent to a Tullock contest. A sponsor invites n R&D firms to carry out an innovative project. Firms submit their products to the designer. The entry of the highest quality wins and its developer is awarded a prize, such as a procurement contract. Each firm i's prize valuation is given by v_i, with $v_1 \geq \cdots \geq v_n > 0$.

Each firm i invests its own input $x_i \geq 0$ to develop the technology. The quality q_i of firm i's product is randomly drawn from a distribution with cumulative distribution function $[\Gamma(q_i)]^{\alpha_i x_i^r}$, with $r \in (0, 1]$ and $\alpha_i \geq 0, \forall i \in \{1, \ldots, n\}$. The function $\Gamma(\cdot)$ is a continuous cumulative distribution function on a support $[\underline{q}, \overline{q}]$, with $\overline{q} > \underline{q}$. By Fullerton and McAfee (1999) and Baye and Hoppe (2003), the term $\alpha_i x_i^r$ can intuitively be interpreted as the number of research ideas generated in developing the product and indicates the firm's research capacity: Each idea allows the firm to produce a prototype, with its quality being drawn from a distribution function $\Gamma(\cdot)$. A firm presents its best prototype to the sponsor as its entry, and its quality follows the distribution function $[\Gamma(q_i)]^{\alpha_i x_i^r}$. The assumption of $r \leq 1$ implies diminishing marginal returns in the development process. By Baye and Hoppe (2003) and Fu and Lu (2012), a firm i wins with a probability[9]

$$\Pr\left(q_i > \max_{j \neq i} q_j\right) = \frac{\alpha_i x_i^r}{\sum_{j=1}^{n} \alpha_j x_j^r},$$

which alludes to the model's isomorphism to a Tullock contest.

The sponsor is endowed with a fixed budget of productive resources – which we normalize to unity – and allocates the resources among firms. We interpret $\alpha_i \in [0, 1]$ as the resource given to a firm $i \in \mathcal{N}$, e.g., the funding provided by the Department of Defense (DoD) to private military contractors, or by major pharmaceutical companies to biomedical startups. Upon receiving α_i, firm i decides on its own input $x_i \geq 0$. The resource α_i can presumably be viewed as a capital input in the development process, while the input x_i can conveniently be interpreted as a labor input sunk by the firm, e.g., the time, energy, and intellectual resources dedicated to the project.

The sponsor chooses allocation plan $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$ to maximize the expected quality of the winning product. Denote by q_{max} the highest quality realized out of all entries. Fixing firms' input profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, it is straightforward to verify that q_{max} is distributed with CDF $[\Gamma(q_{max})]^{\sum_{i=1}^{n} \alpha_i x_i^r}$. Maximizing $\mathbb{E}(q_{max})$ is thus equivalent to maximizing

$$\Lambda := \sum_{i=1}^{n} \alpha_i x_i^r, \tag{11}$$

subject to the equilibrium constraint $x_i = \arg\max_{x_i \geq 0} \pi_i(\boldsymbol{x}, \boldsymbol{\alpha})$, $\alpha_i \geq 0$, and $\sum_{i=1}^{n} \alpha_i \leq 1$, where $\pi_i(\boldsymbol{x}, \boldsymbol{\alpha})$ is firm i's expected payoff.[10] Obviously, the sponsor can effectively exclude a firm by assigning zero resources to it.

It is noteworthy that the resource allocation problem departs from the usual contest design problem based on identity-dependent preferential treatment: The resource α_i not only varies the competitive balance of the competition, but also improves its recipient's actual productivity. The majority of the literature implicitly assumes that identity-dependent treatment is a nominal scoring rule and does not have intrinsic economic value. Technically, maximizing $\sum_{i=1}^{n} \alpha_i x_i^r$ goes beyond the scope of Fu and Wu (2020): $\boldsymbol{\alpha}$ is not only a design instrument, but also a factor in the objective function because it directly accrues to the sponsor's benefit.[11] Further, the power term r enters the objective function (11) nonlinearly.

We can continue to adapt the general approach of Fu and Wu (2020) to this setting despite the complications. The optimization problem can be reformulated similarly, as in Section 2. The correspondence between $\boldsymbol{p}$ and $\boldsymbol{x}$ in equilibrium, (3), continues to hold; the objective function, $\sum_{i=1}^{n} \alpha_i x_i^r$, which contains $\boldsymbol{\alpha}$, also requires that we spell out the relation between $\boldsymbol{\alpha}$ and $\boldsymbol{p}$ in equilibrium. Recall that we use $\mathcal{N}_+(\boldsymbol{p}) \equiv \{i = 1, \ldots, n \mid p_i > 0\}$ to denote the set of active players in equilibrium. The following lemma ensues.

Lemma 2. *Fix any profile of equilibrium winning probabilities $\boldsymbol{p} \equiv (p_1, \ldots, p_n) \in \Delta^{n-1}$ such that $|\mathcal{N}_+(\boldsymbol{p})| \geq 2$. The following resource allocation profile $\boldsymbol{\alpha}(\boldsymbol{p}) \equiv (\alpha_1(\boldsymbol{p}), \ldots, \alpha_n(\boldsymbol{p}))$ uniquely maximizes the expected quality of the winning product out of all of the rules that induce $\boldsymbol{p}$:*

$$\alpha_i(\boldsymbol{p}) = \begin{cases} \frac{(p_i)^{1-r}}{[(1-p_i)v_i]^r} \times \frac{1}{\eta(\boldsymbol{p}, r)} & \text{if} \quad p_i > 0, \\ 0 & \text{if} \quad p_i = 0, \end{cases}$$

where $\eta(\boldsymbol{p}, r) := \sum_{j \in \mathcal{N}_+(\boldsymbol{p})} \left\{(p_j)^{1-r}/[(1-p_j)v_j]^r\right\}$.

We can readily rewrite the objective function as

$$\sum_{i=1}^{n} \alpha_i x_i^r = \sum_{i=1}^{n} \alpha_i \left[p_i(1-p_i)v_i r\right]^r = \frac{r^r}{\eta(\boldsymbol{p}, r)}, \text{ for all } r \in (0, 1].$$

[9] Alternatively, Loury (1979) and Dasgupta and Nti (1998) suggest a patent race model to study "first past the post" R&D contests. Specifically, a number of firms pursue a technological discovery, and a sponsor, to secure the novel technology, rewards the firm that has the earliest success. It can be shown that their model is isomorphic to that of Fullerton and McAfee (1999), and thus the results we derive in this section extend to their setting.

[10] In the research tournament, the sponsor aims to boost the expected quality of the winning product, which is a stochastic output of firms' input $\boldsymbol{x}$; she nevertheless does not focus on the input itself. This design problem differs from maximizing the expected winner's effort – i.e., $\sum_{i=1}^{n} p_i x_i$ – in Fu and Wu (2020).

[11] Fu and Wu (2020) assume a general objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$, which factors in contestants' prize valuations, their effort profiles, and the winning probability distribution, with (2) being one example. Although their general optimization approach provides a guideline for the analysis in this specific setting, their framework does not encompass the scenario in which the designer's welfare depends directly on the design variable $\boldsymbol{\alpha}$, as (11) depicts. Therefore, the general properties of optimal contests established in Fu and Wu (2020) do not immediately extend. The detailed optimization exercise – e.g., optimally selecting active participants – requires a drastically different analysis, as revealed by the proof in the Appendix.

The optimal resource allocation problem boils down to choosing $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$ to *minimize* $\eta(\boldsymbol{p}, r)$, subject to constraints (5). Denote by $\boldsymbol{\alpha}^{**} \equiv (\alpha_1^{**}, \ldots, \alpha_n^{**})$ and $\boldsymbol{p}^{**} \equiv (p_1^{**}, \ldots, p_n^{**})$, respectively, the optimal resource allocation and the equilibrium winning probabilities under the optimal resource allocation. The following can be obtained.

Theorem 2 (*Optimal Resource Allocation in a Research Tournament with Constant Marginal Returns of Resources*)**.** *Suppose that the sponsor aims to maximize the expected quality of the winning product. Fixing* $r \in (0, 1]$*, the equilibrium winning probabilities* $\boldsymbol{p}^{**} \equiv (p_1^{**}, \ldots, p_n^{**})$ *in the optimal contest are given by*

$$p_1^{**} = \frac{\sqrt{\mathcal{M}^2 + 4\chi} + \mathcal{M}}{\sqrt{\mathcal{M}^2 + 4\chi} + \mathcal{M} + 2}, \; p_2^{**} = 1 - p_1^{**}, \quad \text{and } p_3^{**} = \cdots = p_n^{**} = 0,$$

where $\chi := (v_1/v_2)^r \geq 1$ *and* $\mathcal{M} := (\chi - 1)(1 - r)/r$*. Moreover, the corresponding resource allocation* $\boldsymbol{\alpha}^{**} \equiv (\alpha_1^{**}, \ldots, \alpha_n^{**})$ *that induces* $\boldsymbol{p}^{**} \equiv (p_1^{**}, \ldots, p_n^{**})$ *is given by*

$$\alpha_i^{**} = \begin{cases} \dfrac{(p_i^{**})^{1-r}}{[(1-p_i^{**})v_i]^r} \Big/ \left[\dfrac{(p_1^{**})^{1-r}}{[(1-p_1^{**})v_1]^r} + \dfrac{(p_2^{**})^{1-r}}{[(1-p_2^{**})v_2]^r}\right] & \text{if } \; i \in \{1, 2\}, \\ 0 & \text{if } \; i \geq 3. \end{cases}$$

Theorem 2 predicts that the resources will be concentrated on the two strongest firms in the optimum. Our results echo the proposition that restricting entry to two competitors optimizes the tournament in Fullerton and McAfee (1999), but for a different reason. Fullerton and McAfee contend that this "decreases the coordination problem of competing firms and minimizes the duplication of fixed costs." In our setting, in contrast, this avoids wasting costly resources on less productive firms because of the complementarity between the resource α_i and a firm's input x_i. Further, comparing α_1^{**} and α_2^{**} yields the following.

Corollary 3 (*"National Champion" vs. Handicapping*)**.** *Suppose that* $v_1 > v_2$*; then* $\alpha_1^{**} \gtrless \alpha_2^{**}$ *if and only if* $r \lessgtr \frac{1}{2}$.

By Corollary 3, the sponsor may prefer to create a "national champion" by assisting the stronger firm more, which further upsets the balance of the playing field. Favoring the stronger firm will be optimal if and only if the R&D process is sufficiently noisy or risky, i.e., $r < 1/2$. A tension arises between competitive balance and allocative efficiency: On the one hand, favoring the weaker firm enhances competition and increases overall input; on the other hand, it undermines allocative efficiency. Allocative efficiency requires that the resource be entirely concentrated on the firm with the highest input ex post, i.e., the ex ante stronger firm. When the development process involves more risk – i.e., a smaller r – the additional incentive provided by a level playing field would be limited by the lower marginal return to input. Improving allocative efficiency tends to be prioritized more than motivating firms' investment, which compels the sponsor to cultivate a national champion instead of favoring the underdog.

We have assumed that the resource, α_i, has a constant marginal return. This can be interpreted as access to laboratories, equipment, or computing facilities owned by the sponsor. Entitlement to use such resources allows a firm to replicate its experiments and trials. Next, we consider a more generalized model that allows for declining marginal returns for the resource α_i. Specifically, let the production function take the form $f_i(x_i; \alpha_i) = \alpha_i^d \cdot x_i^r$, with $d, r \in (0, 1]$. The sponsor maximizes $\Lambda := \sum_{i=1}^n \alpha_i^d \cdot x_i^r$, in which case the design variable α_i also enters the objective function nonlinearly; the nonlinearity substantially complicates the optimization. Although a full-scale analysis is unavailable, we adapt the approach of Fu and Wu (2020) to obtain the following for the case of identical firms.

Theorem 3 (*Optimal Resource Allocation in a Research Tournament with Diminishing Marginal Returns of Resources*)**.** *Suppose that (i)* $f_i(x_i; \alpha_i) = \alpha_i^d \cdot x_i^r$*, with* $d, r \in (0, 1]$*; (ii) all firms are identical, i.e.,* $v_1 = \cdots = v_n =: v$*; and (iii) the sponsor aims to maximize the expected quality of the winning product. Then all firms that receive a positive amount of resource must win with equal probability. Moreover, the number of active firms, which we denote by* $\kappa^\dagger$*, is given by*

i. *if* $d \in (0, 1 - r]$*, then* $\kappa^\dagger = n$*;*
ii. *if* $d \in (1 - r, 1]$*, then* $\kappa^\dagger \in \left\{\min\left\{n, \left\lfloor 1 + \frac{r}{r+d-1}\right\rfloor\right\}, \min\left\{n, \left\lfloor 2 + \frac{r}{r+d-1}\right\rfloor\right\}\right\}$.

By Theorem 3, more than two firms can be kept active in the optimum when d is sufficiently small. In particular, all firms remain active when d falls below $1 - r$. A smaller d sparks an additional concern for the sponsor: Concentrating the resource on a small subset of firms further decreases marginal return. This tempts the sponsor to spread the resource over more firms, since taking it away from a well-endowed firm would not incur a significant cost. In contrast, transferring it to a poorly endowed firm could yield significant additional gains. This force is in conflict with the effect of complementarity, and the optimum is shaped by the tension. A similar resource allocation problem has been investigated by Fu et al. (2012) in a two-player model. Theorem 2 endogenizes their setting, while Theorem 3 demonstrates the limitations.

4. Conclusion

In this paper, we develop an algorithm adapted from the indirect optimization approach proposed by Fu and Wu (2020) for the optimal design of biased contests in a Tullock setting. We obtain a closed-form solution to the optimum for a wide spectrum of design objectives, which would not be possible under a conventional approach. As previously mentioned, its versatile application is not limited to the design problems described in this paper. Consider, for instance, a context in which the designer aims to maximize the expected winner's effort – i.e., $\sum_{i=1}^n p_i x_i$ – in which case a handy solution can again be achieved using this approach. Our analysis provides useful implications for future research in this vein.

Appendix. Proofs

Proof of Lemma 1. Suppose, to the contrary, that there exists $i, j \in \mathcal{N}$ such that $v_i \geq v_j$ and $p_i^* < p_j^*$ in the optimal contest. It follows immediately that $p_i^* < 1/2$. Next, we show that increasing p_i^* by a sufficiently small $\epsilon > 0$ and decreasing p_j^* by the same amount lead to a strictly higher payoff to the contest designer. With slight abuse of notation, we denote the contest designer's payoff under the alternative winning probabilities by $\Lambda(\epsilon)$. It follows from Eq. (4) that

$$\begin{aligned} \Lambda(\epsilon) = {} & (p_i^* + \epsilon)\left(1 + \frac{\psi}{r} - p_i^* - \epsilon\right) v_i r + (p_j^* - \epsilon) \\ & \times \left(1 + \frac{\psi}{r} - p_j^* + \epsilon\right) v_j r \\ & - \gamma\left[(p_i^* + \epsilon)^2 + (p_j^* - \epsilon)^2\right] + \mathcal{Q}, \end{aligned}$$

where the constant $\mathcal{Q}$ is given by

$$\mathcal{Q} := \sum_{s\in\mathcal{N}\backslash\{i,j\}}\left[p_i\left(1+\frac{\psi}{r}-p_s\right)v_s r\right]-\gamma\sum_{s\in\mathcal{N}\backslash\{i,j\}}(p_s)^2+\frac{\gamma}{n}.$$

Simple algebra yields

$$\begin{aligned}\Lambda'(0) &= \left(1+\frac{\psi}{r}-2p_i^*\right)v_i r-\left(1+\frac{\psi}{r}-2p_j^*\right)v_j r\\ &\quad+2(p_j^*-p_i^*)\gamma\\ &> \left(1+\frac{\psi}{r}-2p_i^*\right)v_i r-\left(1+\frac{\psi}{r}-2p_i^*\right)v_j r\\ &= \left(1+\frac{\psi}{r}-2p_i^*\right)(v_i-v_j)r\geq 0,\end{aligned}$$

where the first inequality follows from the postulated $p_i^* < p_j^*$ and the second inequality follows from $p_i^* < 1/2$. This concludes the proof. ■

Proof of Theorem 1. A sketch analysis is presented in the main text, and it suffices to verify that (i) the equilibrium winning probabilities $\check{\boldsymbol{p}}^m \equiv (\check{p}_1^m, \ldots, \check{p}_n^m)$ specified in (6) solve the auxiliary problem $(\mathcal{P}_m)$ in Step 2 of the algorithm in the main text; and (ii) the constructed weights $\boldsymbol{\alpha}(\boldsymbol{p})$ specified in (8) induce an arbitrary equilibrium winning probability distribution with $p_1 \in (0, 1)$ in Step 5.

We first prove (i). Clearly, $p_{m+1} = \cdots = p_n = 0$ by definition. The first-order condition for the auxiliary problem $(\mathcal{P}_m)$ with respect to p_i yields:

$$\left(1+\frac{\psi}{r}-2p_i\right)v_i r-2\gamma p_i+\beta=0,\text{ for } i=1,\ldots,m,$$

where β is the Lagrange multiplier for the equality constraint $\sum_{i=1}^{n}p_i-1=0$. Rearranging the above condition yields

$$p_i=\frac{v_i(\psi+r)+\beta}{2(v_i r+\gamma)},\text{ for } i=1,\ldots,m. \tag{12}$$

Summing over all of the conditions in (12), we can obtain

$$\beta=\frac{1-\sum_{j=1}^{m}\frac{v_j(\psi+r)}{2(v_j r+\gamma)}}{\sum_{j=1}^{m}\frac{1}{2(v_j r+\gamma)}}. \tag{13}$$

Substituting (13) into (12) yields the following:

$$p_i=\frac{\psi+r}{2r}\left(\frac{v_i r}{v_i r+\gamma}-\frac{1}{v_i r+\gamma}\times\frac{\sum_{j=1}^{m}\frac{v_j r}{v_j r+\gamma}-\frac{2r}{\psi+r}}{\sum_{j=1}^{m}\frac{1}{v_j r+\gamma}}\right),$$
$$\text{for } i=1,\ldots,m,$$

which coincides with the expression $\check{p}_i^m$ specified in (6).

Next, we prove (ii). For $p_i, p_j > 0$, it follows from Eqs. (1) and (3) that

$$\frac{p_i}{p_j}=\frac{\frac{\alpha_i x_i^r}{\sum_{k=1}^{n}\alpha_k x_k^r}}{\frac{\alpha_j x_j^r}{\sum_{k=1}^{n}\alpha_k x_k^r}}=\frac{\alpha_i x_i^r}{\alpha_j x_j^r}=\frac{\alpha_i\left[p_i(1-p_i)v_i r\right]^r}{\alpha_j\left[p_j(1-p_j)v_j r\right]^r},$$

which in turn implies that

$$\frac{\alpha_i}{\alpha_j}=\frac{\frac{(p_i)^{1-r}}{\left[(1-p_i)v_i\right]^r}}{\frac{(p_j)^{1-r}}{\left[(1-p_j)v_j\right]^r}}=\frac{\frac{(p_i)^{1-r}}{[(1-p_i)v_i]^r}\Big/\left[\sum_{k\in\mathcal{N}_+(\boldsymbol{p})}\frac{(p_k)^{1-r}}{[(1-p_k)v_k]^r}\right]}{\frac{(p_j)^{1-r}}{[(1-p_j)v_j]^r}\Big/\left[\sum_{k\in\mathcal{N}_+(\boldsymbol{p})}\frac{(p_k)^{1-r}}{[(1-p_k)v_k]^r}\right]}=\frac{\alpha_i(\boldsymbol{p})}{\alpha_j(\boldsymbol{p})}.$$

This concludes the proof. ■

Proof of Corollary 1. For notational convenience, define

$$\bar{v}_m := \frac{\sum_{j=1}^{m}\frac{v_j}{v_j r+\gamma}}{\sum_{j=1}^{m}\frac{1}{v_j r+\gamma}},\text{ and }\bar{\bar{v}}_m := \frac{\sum_{j=1}^{m}\frac{v_j}{(v_j+\frac{\gamma_H}{r})(v_j+\frac{\gamma_L}{r})}}{\sum_{j=1}^{m}\frac{1}{(v_j+\frac{\gamma_H}{r})(v_j+\frac{\gamma_L}{r})}},$$

for $m = 1, \ldots, n$.

It is useful to state an intermediate result.

Lemma 3. *The following statements hold:*

i. *Fix $\gamma \geq 0$ and suppose that $\psi_H > \psi_L \geq 0$, then for all $m \in \mathcal{N}$, $\check{p}_i^m(\psi_H, \gamma) \geq \check{p}_i^m(\psi_L, \gamma)$ if and only if $v_i \geq \bar{v}_m$.*
ii. *Fix $\psi \geq 0$ and suppose that $\gamma_H > \gamma_L \geq 0$, then for all $m \in \mathcal{N}$, $\check{p}_i^m(\psi, \gamma_H) \geq \check{p}_i^m(\psi, \gamma_L)$ if and only if $v_i \leq \bar{\bar{v}}_m$.*

Proof. The first part of the lemma is obvious for $m = 1$ and it remains to prove the result for the case $m \geq 2$. Recall that

$$\begin{aligned}\check{p}_i^m &= \frac{\psi+r}{2}\frac{v_i-\frac{\sum_{j=1}^{m}\frac{v_j}{v_j r+\gamma}-\frac{2}{\psi+r}}{\sum_{j=1}^{m}\frac{1}{v_j r+\gamma}}}{v_i r+\gamma}\\ &= \frac{\psi+r}{2}\frac{1}{v_i r+\gamma}\left(v_i-\frac{\sum_{j=1}^{m}\frac{v_j}{v_j r+\gamma}}{\sum_{j=1}^{m}\frac{1}{v_j r+\gamma}}\right)+\frac{1}{(v_i r+\gamma)\sum_{j=1}^{m}\frac{1}{v_j r+\gamma}}.\end{aligned}$$

From the above equation, we can obtain that

$$\check{p}_i^m(\psi_H,\gamma)-\check{p}_i^m(\psi_L,\gamma)\gtreqless 0\Leftrightarrow v_i\gtreqless\frac{\sum_{j=1}^{m}\frac{v_j}{v_j r+\gamma}}{\sum_{j=1}^{m}\frac{1}{v_j r+\gamma}}\equiv\bar{v}_m.$$

Similarly, we prove the second part of the lemma. Note that $\check{p}_i^m$ can be rewritten as follows:

$$\begin{aligned}\check{p}_i^m &= \frac{\psi+r}{2r}\frac{v_i-\frac{\sum_{j=1}^{m}\frac{v_j}{v_j+\tilde{\gamma}}-\frac{2r}{\psi+r}}{\sum_{j=1}^{m}\frac{1}{v_j+\tilde{\gamma}}}}{v_i+\tilde{\gamma}}\\ &= \frac{\psi+r}{2r}\left(1-\frac{\frac{\sum_{j=1}^{m}\frac{v_j}{v_j+\tilde{\gamma}}-\frac{2r}{\psi+r}}{\sum_{j=1}^{m}\frac{1}{v_j+\tilde{\gamma}}}+\tilde{\gamma}}{v_i+\tilde{\gamma}}\right)\\ &= \frac{\psi+r}{2r}\left(1-\frac{m-\frac{2r}{\psi+r}}{(v_i+\tilde{\gamma})\sum_{j=1}^{m}\frac{1}{v_j+\tilde{\gamma}}}\right),\end{aligned}$$

where $\tilde{\gamma} := \gamma/r$. It is straightforward to see that

$$\begin{aligned}\check{p}_i^m(\psi,\gamma_H)-\check{p}_i^m(\psi,\gamma_L) &= \frac{\psi+r}{2r}\left(m-\frac{2r}{\psi+r}\right)\\ &\quad\times\frac{(v_i+\tilde{\gamma}_H)\sum_{j=1}^{m}\frac{1}{v_j+\tilde{\gamma}_H}-(v_i+\tilde{\gamma}_L)\sum_{j=1}^{m}\frac{1}{v_j+\tilde{\gamma}_L}}{(v_i+\tilde{\gamma}_H)\sum_{j=1}^{m}\frac{1}{v_j+\tilde{\gamma}_H}(v_i+\tilde{\gamma}_L)\sum_{j=1}^{m}\frac{1}{v_j+\tilde{\gamma}_L}}\\ &= \frac{\psi+r}{2r}\left(m-\frac{2r}{\psi+r}\right)\\ &\quad\times\frac{\sum_{j=1}^{m}\frac{v_j-v_i}{(v_j+\tilde{\gamma}_H)(v_j+\tilde{\gamma}_L)}(\tilde{\gamma}_H-\tilde{\gamma}_L)}{(v_i+\tilde{\gamma}_H)\sum_{j=1}^{m}\frac{1}{v_j+\tilde{\gamma}_H}(v_i+\tilde{\gamma}_L)\sum_{j=1}^{m}\frac{1}{v_j+\tilde{\gamma}_L}}.\end{aligned}$$

Therefore, we can obtain that

$$\begin{aligned}\check{p}_i^m(\psi,\gamma_H)-\check{p}_i^m(\psi,\gamma_L)\gtreqless 0 &\Leftrightarrow \sum_{j=1}^{m}\frac{v_j-v_i}{(v_j+\tilde{\gamma}_H)(v_j+\tilde{\gamma}_L)}\gtreqless 0\\ &\Leftrightarrow v_i\lesseqgtr\frac{\sum_{j=1}^{m}\frac{v_j}{(v_j+\tilde{\gamma}_H)(v_j+\tilde{\gamma}_L)}}{\sum_{j=1}^{m}\frac{1}{(v_j+\tilde{\gamma}_H)(v_j+\tilde{\gamma}_L)}}\equiv\bar{\bar{v}}_m.\end{aligned}$$ ■

Corollary 1(i) follows immediately from $\kappa(\psi_H,\gamma)=\kappa(\psi_L,\gamma)$ and Lemma 3(i). Similarly, Corollary 1(ii) follows from $\kappa(\psi,\gamma_H)=\kappa(\psi,\gamma_L)$ and Lemma 3(ii). This concludes the proof. ■

Proof of Corollary 2. Part i follows immediately from the fact that $[\sum_{j=1}^{m}\frac{v_j r}{v_j r+\gamma}-\frac{2r}{\psi+r}]/[\sum_{j=1}^{m}\frac{1}{v_j r+\gamma}]$ is strictly increasing in ψ, and it remains to show part ii. Carrying out the algebra, $\kappa(\psi,\gamma)$ in Eq. (7) can be written as

$$\kappa(\psi,\gamma):=\max\left\{m=1,\ldots,n\;\middle|\;\sum_{j=1}^{m}\frac{v_j-v_m}{v_j r+\gamma}<\frac{2}{\psi+r}\right\}.$$

Define $\mathcal{H}(m,\gamma)$ as

$$\mathcal{H}(m,\gamma):=\sum_{j=1}^{m}\frac{v_j-v_m}{v_j r+\gamma}.$$

It follows immediately that

$$\begin{aligned}\mathcal{H}(m+1,\gamma)-\mathcal{H}(m,\gamma)&=\sum_{j=1}^{m+1}\frac{v_j-v_{m+1}}{v_j r+\gamma}-\sum_{j=1}^{m}\frac{v_j-v_m}{v_j r+\gamma}\\&=\sum_{j=1}^{m}\frac{v_j-v_{m+1}}{v_j r+\gamma}-\sum_{j=1}^{m}\frac{v_j-v_m}{v_j r+\gamma}\\&=\sum_{j=1}^{m}\frac{v_m-v_{m+1}}{v_j r+\gamma}\geq 0.\end{aligned}$$

Therefore, $\mathcal{H}(m,\gamma)$ is weakly increasing in m. Moreover, it is straightforward to see that $\mathcal{H}(m,\gamma)$ is weakly decreasing in γ. These two facts imply instantly that $\kappa(\psi,\gamma)$ is weakly increasing in γ, holding fixed $\psi\geq 0$. This completes the proof. ■

Proof of Lemma 2. The proof is similar to that of Step 5 of the algorithm that leads to Theorem 1, and is omitted for brevity. ■

Proof of Theorem 2. We first show that only the two strongest firms would remain active in the optimal contest. Similar to Lemma 1, we can show that $p_1^{**}\geq\cdots\geq p_n^{**}$ in the optimal contest. We consider the following two cases depending on r relative to one.

Case I: $r=1$. Consider the following sequence of auxiliary problems $(\widehat{\mathcal{P}}_m)$: For each $m=2,\ldots,n$, the sponsor minimizes

$$\sum_{j=1}^{m}\frac{1}{(1-p_j)v_j},$$

subject to the plausibility constraint $\sum_{i=1}^{n}p_i=1$, ignoring the nonnegativity constraint $p_i\geq 0$ for $i\in\{1,\ldots,m\}$ and setting $p_i=0$ for $i\in\mathcal{N}\setminus\{1,\ldots,m\}$. The solution to the auxiliary optimization problem $(\widehat{\mathcal{P}}_m)$, which we denote by $\ddot{\boldsymbol{p}}^m\equiv(\ddot{p}_1^m,\ldots,\ddot{p}_n^m)$, can be solved explicitly by computing the first-order conditions, and is given by

$$\ddot{p}_i^m=\begin{cases}1-\frac{1}{\sqrt{v_i}}\times\frac{m-1}{\sum_{j=1}^{m}\frac{1}{\sqrt{v_j}}} & \text{if}\quad i\in\{1,\ldots,m\},\\ 0 & \text{if}\quad i\in\mathcal{N}\setminus\{1,\ldots,m\}.\end{cases}$$

The corresponding η can be derived as the following:

$$\eta(\ddot{\boldsymbol{p}}^m,1)=\frac{\left(\sum_{i=1}^{m}\frac{1}{\sqrt{v_i}}\right)^2}{m-1}.$$

Next, we show that $\eta(\ddot{\boldsymbol{p}}^m,1)<\eta(\ddot{\boldsymbol{p}}^{m+1},1)$ for all $m\in\{2,\ldots,n-1\}$, which is equivalent to

$$\frac{\sum_{i=1}^{m}\frac{1}{\sqrt{v_i}}}{\frac{1}{\sqrt{v_{m+1}}}+\sum_{i=1}^{m}\frac{1}{\sqrt{v_i}}}<\frac{\sqrt{m-1}}{\sqrt{m}}.$$

Note that $\sum_{i=1}^{m}\frac{1}{\sqrt{v_i}}\leq\frac{m}{\sqrt{v_{m+1}}}$ due to the postulated $v_1\geq\cdots\geq v_n$. Therefore, we have that

$$\frac{\sum_{i=1}^{m}\frac{1}{\sqrt{v_i}}}{\frac{1}{\sqrt{v_{m+1}}}+\sum_{i=1}^{m}\frac{1}{\sqrt{v_i}}}\leq\frac{m}{m+1},$$

and it remains to prove $\frac{m}{m+1}<\frac{\sqrt{m-1}}{\sqrt{m}}$, which can easily be shown to hold after some algebra. Further, it is straightforward to verify that $\ddot{p}_2^2>0$. Therefore, only the two strongest firms would remain active in the optimum for the case $r=1$.

Case II: $r<1$. It is useful to state an intermediate result.

Lemma 4. *Suppose $0<\ell<\frac{2}{3}$, $\mu\geq 1$ and $p\in(0,\ell)$, then*

$$\frac{(p)^{1-r}}{(1-p)^r}+\mu\times\frac{(\ell-p)^{1-r}}{(1-\ell+p)^r}>\frac{\ell^{1-r}}{(1-\ell)^r}.$$

Proof. Define $\mathcal{G}(\ell,p)$ as

$$\mathcal{G}(\ell,p):=\frac{(p)^{1-r}}{(1-p)^r}+\mu\times\frac{(\ell-p)^{1-r}}{(1-\ell+p)^r}-\frac{\ell^{1-r}}{(1-\ell)^r}.$$

We want to show $\mathcal{G}(\ell,p)>0$ for $\ell>p$. Fixing p, let us view $\mathcal{G}(\cdot)$ as a function of ℓ. Clearly, we have that $\mathcal{G}(p,p)=0$. Moreover, we have that

$$\begin{aligned}\mathcal{G}\left(\frac{2}{3},p\right)&=\frac{(p)^{1-r}}{(1-p)^r}+\mu\times\frac{\left(\frac{2}{3}-p\right)^{1-r}}{\left(\frac{1}{3}+p\right)^r}-\frac{\left(\frac{2}{3}\right)^{1-r}}{\left(\frac{1}{3}\right)^r}\\&\geq\frac{(p)^{1-r}}{(1-p)^r}+\frac{\left(\frac{2}{3}-p\right)^{1-r}}{\left(\frac{1}{3}+p\right)^r}-\frac{\left(\frac{2}{3}\right)^{1-r}}{\left(\frac{1}{3}\right)^r},\end{aligned}$$

where the inequality follows from the postulated $\mu\geq 1$. It can be verified that

$$\frac{(p)^{1-r}}{(1-p)^r}+\frac{\left(\frac{2}{3}-p\right)^{1-r}}{\left(\frac{1}{3}+p\right)^r}-\frac{\left(\frac{2}{3}\right)^{1-r}}{\left(\frac{1}{3}\right)^r}\geq 0,\forall(p,r)\in\left[0,\frac{2}{3}\right]\times(0,1).$$

Therefore, to prove the lemma, it suffices to show that $\mathcal{G}(\ell,p)$ is single-peaked or increasing in ℓ.

Carrying out the algebra, we have that

$$\begin{aligned}\frac{\partial\mathcal{G}(\ell,p)}{\partial\ell}&=\mu\times\frac{(1-\ell+p)-(1-2\ell+2p)r}{(\ell-p)^r(1-\ell+p)^{1+r}}\\&\quad-\frac{(1-\ell)-(1-2\ell)r}{\ell^r(1-\ell)^{1+r}}.\end{aligned}$$

It can be verified that $\frac{\partial\mathcal{G}(\ell,p)}{\partial\ell}>0$ is equivalent to

$$\begin{aligned}\mathcal{Z}(\ell,p,r):=&\log(\mu)+\log\left((1-\ell+p)-(1-2\ell+2p)r\right)\\&-\log\left((1-\ell)-(1-2\ell)r\right)\\&-r\log\left(\frac{\ell-p}{\ell}\right)-(1+r)\log\left(\frac{1-\ell+p}{1-\ell}\right)>0.\end{aligned}$$

Note that $\mathcal{Z}(p,p,r)=\infty$. To prove that $\mathcal{G}(\ell,p)$ is single-peaked or increasing in ℓ, it suffices to show that $\mathcal{Z}(\ell,p,r)$ is strictly decreasing in ℓ, that is,

$$\begin{aligned}\frac{\partial\mathcal{Z}(\ell,p,r)}{\partial\ell}=&\frac{2r-1}{(1-\ell+p)-(1-2\ell+2p)r}\\&-\frac{2r-1}{(1-\ell)-(1-2\ell)r}+r\left(\frac{1}{\ell}-\frac{1}{\ell-p}\right)\end{aligned}$$

$$+ (1+r)\left(\frac{1}{1-\ell+p}-\frac{1}{1-\ell}\right)<0,$$
$$\forall(\ell,p,r)\in(0,2/3)\times(0,\ell)\times(0,1).$$

Carrying out the algebra, $\frac{\partial \mathcal{Z}(\ell,p,r)}{\partial \ell}<0$ is equivalent to

$$\mathcal{W}(r):=\frac{(2r-1)^2}{[(1-\ell+p)-(1-2\ell+2p)r]\times[(1-\ell)-(1-2\ell)r]}-\frac{r}{(\ell-p)\ell}-\frac{1+r}{(1-\ell+p)(1-\ell)}<0.$$

For $r\in(0,\frac{1}{2}]$, $\mathcal{W}(r)$ can be bounded above by

$$\begin{aligned}\mathcal{W}(r)&<\frac{(2r-1)^2}{[(1-\ell+p)-(1-2\ell+2p)r]\times[(1-\ell)-(1-2\ell)r]}-\frac{1}{(1-\ell+p)(1-\ell)}\\&\leq\frac{(1-2r)^2}{(1-\ell+p)(1-\ell)(1-r)^2}-\frac{1}{(1-\ell+p)(1-\ell)}\\&=\frac{1}{(1-\ell+p)(1-\ell)}\times\left[\frac{(1-2r)^2}{(1-r)^2}-1\right]<0,\end{aligned}$$

where the first strict inequality follows from $r>0$; the second inequality follows from $(1-\ell)-(1-2\ell)r=(1-\ell)(1-r)+\ell r\geq(1-\ell)(1-r)$ and $(1-\ell+p)-(1-2\ell+2p)r=(1-\ell+p)(1-r)+(\ell-p)r\geq(1-\ell+p)(1-r)$; and the last inequality follows from $\frac{1-2r}{1-r}<1$.

Similarly, for $r\in(\frac{1}{2},1)$, we have that

$$\begin{aligned}\mathcal{W}(r)&\leq\frac{(2r-1)^2}{(\ell-p)\ell r^2}-\frac{r}{(\ell-p)\ell}-\frac{1+r}{(1-\ell+p)(1-\ell)}\\&=\frac{1}{(\ell-p)\ell}\times\frac{1-r}{r^2}\times\left(r^2-3r+1\right)-\frac{1+r}{(1-\ell+p)(1-\ell)}\\&<0,\end{aligned}$$

where the first inequality follows from $(1-\ell)-(1-2\ell)r=(1-\ell)(1-r)+\ell r\geq\ell r$ and $(1-\ell+p)-(1-2\ell+2p)r=(1-\ell+p)(1-r)+(\ell-p)r\geq(\ell-p)r$; and the second inequality follows from $r^2-3r+1<0$ for $r\in[\frac{1}{2},1)$. This completes the proof. ■

By Lemma 4, if $p_i^{**}>0$ and $p_j^{**}>0$, then we must have that $p_i^{**}+p_j^{**}\geq\frac{2}{3}$. Therefore, there are at most three active firms in the optimum. Furthermore, when three firms remain active, we must have that $p_1^{**}=p_2^{**}=p_3^{**}=\frac{1}{3}$, which can easily be proved to be suboptimal. Therefore, only the two strongest firms would remain active in the optimum.

Next, we characterize the optimal equilibrium winning probabilities $\boldsymbol{p}^{**}\equiv(p_1^{**},\ldots,p_n^{**})$. Because $p_i^{**}=0$ for $i\in\{3,\ldots,n\}$, we must have $p_2^{**}=1-p_1^{**}$. Therefore, the sponsor's optimization problem can be simplified as

$$\min_{p_1\in(0,1)}\frac{(p_1)^{1-r}}{(1-p_1)^r}+\chi\frac{(1-p_1)^{1-r}}{(p_1)^r},$$

where $\chi:=(v_1/v_2)^r\geq1$. The first-order condition with respect to p_1 yields

$$\frac{p_1}{1-p_1}-\chi\frac{1-p_1}{p_1}=\frac{(\chi-1)(1-r)}{r}=:\mathcal{M},$$

from which p_1^{**} can be solved for as

$$p_1^{**}=\frac{\sqrt{\mathcal{M}^2+4\chi}+\mathcal{M}}{\sqrt{\mathcal{M}^2+4\chi}+\mathcal{M}+2}.$$

This completes the proof. ■

Proof of Corollary 3. It is straightforward to verify that $\alpha_1>\alpha_2$ if and only if $p_1^{**}>\frac{\chi}{\chi+1}$, which can be further simplified as

$$\mathcal{M}>\chi-1\Leftrightarrow\frac{(\chi-1)(1-r)}{r}>\chi-1\Leftrightarrow r<\frac{1}{2}.$$

This completes the proof. ■

Proof of Theorem 3. Similar to Lemma 2, fixing the equilibrium winning probabilities $\boldsymbol{p}\equiv(p_1,\ldots,p_n)$, we can show that

$$x_i=p_i(1-p_i)v_ir,$$

and

$$\alpha_i(\boldsymbol{p})=\left(\frac{(p_i)^{1-r}}{\left[(1-p_i)\,v_i\right]^r}\right)^{\frac{1}{d}}\Bigg/\sum_{j\in\mathcal{N}_+(\boldsymbol{p})}\left(\frac{(p_j)^{1-r}}{\left[(1-p_j)\,v_j\right]^r}\right)^{\frac{1}{d}}.$$

Carrying out the algebra, we can obtain that

$$\begin{aligned}\sum_{i=1}^n\alpha_i^d\cdot x_i^r&=\sum_{i=1}^n\alpha_i^d\left[p_i(1-p_i)v_ir\right]^r\\&=\frac{r^r}{\left[\sum_{j\in\mathcal{N}_+(\boldsymbol{p})}\left(\frac{(p_j)^{1-r}}{\left[(1-p_j)v_j\right]^r}\right)^{\frac{1}{d}}\right]^d},\forall d,r\in(0,1].\end{aligned}$$

Fix $r,d\in(0,1]$, define $g(\cdot)$ as

$$g(p):=\left[\frac{p^{1-r}}{(1-p)^r}\right]^{\frac{1}{d}}.$$

When firms are identical (i.e., $v_1=\cdots=v_n$), the designer's optimization problem is equivalent to choosing $\boldsymbol{p}\equiv(p_1,\ldots,p_n)$ to minimize

$$\sum_{j\in\mathcal{N}_+(\boldsymbol{p})}g(p_j),$$

subject to constraints (5). To proceed, it is useful to prove several intermediate results.

Lemma 5. *Fix $r,d\in(0,1]$. The following statements hold:*

i. *If $d+r\leq1$, then $g(p)$ is strictly convex in p for all $p\in[0,1)$.*
ii. *If $d+r>1$, then there exists a threshold $\tilde{p}\in(0,1)$ such that $g(p)$ is strictly concave on $[0,\tilde{p}]$ and is strictly convex on $[\tilde{p},1)$. Moreover, $g'''(p)>0$ for all $p\in(0,1)$.*

Proof. Carrying out the algebra, we can obtain that

$$g'(p)=\frac{g(p)}{d}\left(\frac{1-r}{p}+\frac{r}{1-p}\right)>0,$$

and

$$g''(p)=\frac{g(p)}{d^2(1-p)^2p^2}\mathcal{S}(p),$$

where $\mathcal{S}(p)$ is defined as

$$\mathcal{S}(p):=(1-d-2r)\left[(1-2r)p^2-2(1-r)p\right]+(1-r)(1-d-r).\tag{14}$$

It is obvious that $\mathcal{S}(\cdot)$ is quadratic in p for $r\neq1/2$ and is linear in p for $r=1/2$, and it can be verified that $\mathcal{S}(p)$ is monotone on $(0,1)$ for all $r\in(0,1]$. Next, note that $\mathcal{S}(0)=(1-r)(1-d-r)$ and $\mathcal{S}(1)=r(d+r)>0$. Therefore, if $d+r\leq1$, then $\mathcal{S}(p)>0$ for all $p\in(0,1)$. If $d+r>1$, there exists $\tilde{p}\in(0,1)$ such that $\mathcal{S}(p)<0$ on $(0,\tilde{p})$ and $\mathcal{S}(p)>0$ on $(\tilde{p},1)$.

It remains to show that $g'''(p) > 0$ for all $p \in (0, 1)$ when $d + r > 1$. For $r = 1$, we have that

$$g'''(p) = \frac{g(p)(d+1)(2d+1)}{d^3(1-p)^3} > 0, \forall p \in (0, 1).$$

For $r \neq 1$, we have that

$$g'''(p) = \underbrace{\frac{g(p)(1-r)(d+2r-1)(2d+2r-1)}{d^3(1-p)^3 p^3}}_{>0} \times \left[\frac{2r-1}{1-r}p^3 + 3p^2 - 3\frac{w}{1+w}p + \frac{w\left(2w+\frac{1-r}{r}\right)}{(1+w)\left(2w+\frac{1}{r}\right)}\right],$$

where $w := (d + r - 1)/r \in (0, 1]$. Note that

$$\begin{aligned}&\frac{2r-1}{1-r}p^3 + 3p^2 - 3\frac{w}{1+w}p + \frac{w\left(2w+\frac{1-r}{r}\right)}{(1+w)\left(2w+\frac{1}{r}\right)}\\ &= \underbrace{p^2(1-p) + \frac{r}{1-r}p^3}_{\geq 0} + 2p^2 - 3\frac{w}{1+w}p + \underbrace{\frac{w\left(2w+\frac{1-r}{r}\right)}{(1+w)\left(2w+\frac{1}{r}\right)}}_{\geq \frac{w}{1+w}\frac{2w}{2w+1}}\\ &\geq 2p^2 - 3\frac{w}{1+w}p + \frac{w}{1+w}\frac{2w}{2w+1}\\ &\geq \frac{w^2(7-2w)}{8(w+1)^2(2w+1)} > 0.\end{aligned}$$

Therefore, $g'''(p) > 0$ for all $p \in (0, 1)$ and $r \in (1 - d, 1]$. This concludes the proof. ■

Lemma 6. *Suppose that $d + r > 1$ and $p \in (0, 1)$. Then $g(p) \gtreqless 2g(p/2)$ if and only if $p \gtreqless p^\star$, where $p^\star := 2\left(1 - \frac{1}{2 - 2^{-\frac{r+d-1}{r}}}\right)$. Moreover, $p^\star/2 > \tilde{p}$, where $\tilde{p}$ is defined in Lemma 5.*

Proof. The first part of the lemma is trivial, and it remains to prove the second part. Recall that we denote $(r + d - 1)/r$ by w in the proof of Lemma 5. It follows from Eq. (14) that

$$\begin{aligned}\mathcal{S}\left(\frac{p^\star}{2}\right) &= \frac{-2^{\frac{d+1}{r}+2}(d+1)r + 4^{\frac{1}{r}}r(d+r) + 4^{\frac{d+r}{r}}[(2r-1)d+1]}{\left(2^{\frac{d}{r}+2} - 2^{\frac{1}{r}}\right)^2}\\ &= \frac{4^{\frac{1}{r}}r}{\left(2^{\frac{d}{r}+2} - 2^{\frac{1}{r}}\right)^2}\left[rw + 1 + (3-w)4^w \right.\\ &\quad \left. -2(2 + rw - r)2^w\right].\end{aligned}$$

Define $\mathcal{T}(r, w) := rw + 1 + (3 - w)4^w - 2(2 + rw - r)2^w$. It can be verified that $\partial\mathcal{T}/\partial r > 0$, and thus $\mathcal{T}(r, w) > \mathcal{T}(0, w) = 1 + (3 - w)4^w - 2^{w+2} > 0$, which in turn implies that $\mathcal{S}(p^\star/2) > 0$ and $g''(p^\star/2) > 0$. Therefore, we must have that $p^\star/2 > \tilde{p}$. This concludes the proof. ■

Lemma 7. *Suppose that $d + r > 1$ and $0 \leq a \leq b \leq \tilde{p}$, where $\tilde{p}$ is defined in Lemma 5. Then $g(a) + g(b) \geq g(a + b)$.*

Proof. Define $\xi(p)$ as

$$\xi(p) := g(p) - g(p - \tilde{p}) - g(\tilde{p}), p \in [\tilde{p}, 2\tilde{p}].$$

Note that $g(0) = 0$ and thus $\xi(\tilde{p}) = 0$. By Lemma 6, $\xi(2\tilde{p}) = g(2\tilde{p}) - 2g(\tilde{p}) < 0$. Moreover, $g''(p)$ is increasing in p by Lemma 5, and thus $\xi''(p) = g''(p) - g''(p - \tilde{p}) \geq 0$, which in turn implies that $\xi(p)$ is convex in p for all $p \in [\tilde{p}, 2\tilde{p}]$. Therefore, we have that

$$\xi(p) \leq 0, \forall p \in [\tilde{p}, 2\tilde{p}]. \tag{15}$$

We can now prove the lemma. If $a+b \leq \tilde{p}$, then $g(\cdot)$ is concave in p for $p \in [a, b]$ by Lemma 5 and thus $g(a) + g(b) \geq g(a + b)$. If $a + b > \tilde{p}$, then $\xi(a + b) \leq 0$ by (15), which is equivalent to

$$g(a + b - \tilde{p}) + g(\tilde{p}) \geq g(a + b).$$

Moreover, because $g(\cdot)$ is concave in p for $p \in [0, \tilde{p}]$ by Lemma 5, we have that

$$g(a) + g(b) \geq g(a + b - \tilde{p}) + g(\tilde{p}).$$

Summing the above two inequalities, we immediately obtain that $g(a) + g(b) \geq g(a + b)$. This concludes the proof. ■

Lemma 8. *Suppose that $d + r > 1$. Fixing $a \in [0, \tilde{p})$, there exists a unique threshold $\hat{p}(a)$ that lies in $(\tilde{p}, 1)$, such that*

$$g(a) + g(p) \gtreqless 2g\left(\frac{a+p}{2}\right) \Leftrightarrow p \gtreqless \hat{p}(a).$$

Proof. Fix $a \in [0, \tilde{p})$. Let us define

$$\lambda(a, p) := g(a) + g(p) - 2g\left(\frac{a+p}{2}\right), \text{ for } p \in [0, 1).$$

We first prove existence. Recall that $g(\cdot)$ is strictly concave in p for $p \in [0, \tilde{p}]$. Therefore, we have that

$$\lambda(a, p) = 2\left[\frac{g(a) + g(p)}{2} - g\left(\frac{a+p}{2}\right)\right] < 0, \forall p \in [0, \tilde{p}].$$

Moreover, it is straightforward to verify that

$$\lim_{p\nearrow 1} \lambda(a, p) = \infty > 0.$$

Therefore, there exists at least one solution to $\lambda(a, p) = 0$, and all solutions must be strictly greater than $\tilde{p}$.

Next, we show that $\lambda(a, p) = 0$ has a unique solution. It suffices to show that if $\lambda(a, \hat{p}) = 0$ for some $\hat{p}$, then we must have $\partial\lambda(a, p)/\partial p|_{p=\hat{p}} > 0$, which is equivalent to

$$g'(\hat{p}) > g'\left(\frac{a+\hat{p}}{2}\right).$$

Suppose, to the contrary, that $g'(\hat{p}) \leq g'\left((a + \hat{p})/2\right)$. Because $g(\cdot)$ is convex in p for $p \in [\tilde{p}, 1)$ by Lemma 5, we must have that $(a + \hat{p})/2 < \tilde{p}$, which implies that $g'(\cdot)$ first decreases and then increases with p for $p \in \left(\frac{a+\hat{p}}{2}, \hat{p}\right)$. Therefore, we have that

$$g'(p) < \max\left\{g'(\hat{p}), g'\left(\frac{a+\hat{p}}{2}\right)\right\} = g'\left(\frac{a+\hat{p}}{2}\right), \forall p \in \left(\frac{a+\hat{p}}{2}, \hat{p}\right). \tag{16}$$

Similarly, it follows from $(a + \hat{p})/2 < \tilde{p}$ and Lemma 5 that $g(\cdot)$ is concave on $\left(0, \frac{a+\hat{p}}{2}\right)$. Therefore, we have that

$$g'(p) > g'\left(\frac{a+\hat{p}}{2}\right), \forall p \in \left(a, \frac{a+\hat{p}}{2}\right). \tag{17}$$

Next, by $\lambda(a, \hat{p}) = 0$ and the Mean Value Theorem, there exists $(z_1, z_2) \in \left(a, \frac{a+\hat{p}}{2}\right) \times \left(\frac{a+\hat{p}}{2}, \hat{p}\right)$ such that

$$g'(z_1) = \frac{g\left(\frac{a+\hat{p}}{2}\right) - g(a)}{\frac{\hat{p}-a}{2}} = \frac{g(\hat{p}) - g\left(\frac{a+\hat{p}}{2}\right)}{\frac{\hat{p}-a}{2}} = g'(z_2).$$

However, the above equality cannot hold because

$$g'(z_1) > g'\left(\frac{a+\hat{p}}{2}\right) > g'(z_2),$$

where the first inequality follows from (17) and the second inequality follows from (16). A contradiction. This concludes the proof. ■

Lemma 9. *Suppose that $d + r > 1$. If there exists $(a, b) \in (0, \tilde{p}) \times (\tilde{p}, 1)$ such that $g(a) + g(b) = 2g\left(\frac{a+b}{2}\right)$, then $a + b \le p^\star$, where $p^\star$ is defined in Lemma 6.*

Proof. Fixing $a \in (0, \tilde{p})$, it follows immediately from Lemma 8 that $b = \hat{p}(a)$. Define $m(a)$ as

$$m(a) := \frac{a + \hat{p}(a)}{2}, \text{ for } a \in [0, \tilde{p}).$$

It suffices to show that $m(a) = [a + \hat{p}(a)]/2 \le p^\star/2$. Note that $\lim_{a \nearrow \tilde{p}} \hat{p}(a) = \tilde{p}$; together with Lemma 6, we have that

$$\lim_{a \nearrow \tilde{p}} m(a) = \tilde{p} < \frac{p^\star}{2}.$$

Suppose, to the contrary, that there exists some $a' \in [0, \tilde{p})$ such that $m(a') > p^\star/2$, then there must exist $a'' \in (p', \tilde{p})$ such that $m(a'') = p^\star/2$. Denote $\hat{p}(a'')$ by b''. Then $a'' + b''$ and $g(a'') + g(b'') = 2g\left(\frac{a''+b''}{2}\right)$ by definition. To prove the lemma, it suffices to show that there exists no $a, b > 0$, with $a \ne b$, such that $a + b = p^\star$ and $g(a) + g(b) = 2g\left(\frac{a+b}{2}\right)$.

Define $\zeta(\cdot)$ as

$$\zeta(a) := g(a) + g(p^\star - a) - 2g\left(\frac{p^\star}{2}\right), \text{ for } \left[0, \frac{p^\star}{2}\right].$$

It remains to show that $\zeta(a) = 0$ has no solution on $\left(0, \frac{p^\star}{2}\right)$. Note that $\zeta(0) = 0$ and $\zeta(p^\star/2) = 0$. Moreover, $\zeta'(0) = +\infty$ and $\zeta'(p^\star/2) = 0$. Next, we show that $\zeta'(a) = 0$ has a unique solution on the interval $\left(0, \frac{p^\star}{2}\right)$, from which we can conclude that $\zeta(a)$ follows an inverted-U-shaped curve on $\left(0, \frac{p^\star}{2}\right)$ and thus $\zeta(a) = 0$ has no solution on $\left(0, \frac{p^\star}{2}\right)$.

Note that a solution to $\zeta'(a) = 0$ must exist from the Mean Value Theorem and the fact that $\zeta(0) = \zeta(p^\star/2)$. Moreover, it is straightforward to verify that $\zeta'(a) = 0$ is equivalent to $g'(a) = g'(p^\star - a)$. Therefore, it suffices to show that $g'(a) = g'(p^\star - a)$ has a unique solution on $\left(0, \frac{p^\star}{2}\right)$. In what follows, we prove the result for the case $r \ne 0.5$. The analysis is similar for the case $r = 0.5$, and is omitted for brevity.

Carrying out the algebra, $g'(a) = g'(p^\star - a)$ is equivalent to

$$\tau(a) := \log(c_1 - a) - \log(a + c_1 - p^\star) + c_2 \log(p^\star - a) - c_2 \log a + c_3 \log(a + 1 - p^\star) - c_3 \log(1 - a) = 0,$$

where $c_1 := -(1-r)/(2r-1) \in (-\infty, 0] \cup (1, \infty)$, $c_2 := (d + r - 1)/d \in (0, r]$, and $c_3 := (d + r)/d \ge 1 + c_2 > 0$.

Taking the derivative of $\tau(\cdot)$ with respect to a yields

$$\tau'(a) = \frac{c_3(2 - p^\star)}{(1-a)(a + 1 - p^\star)} - \frac{c_2 p^\star}{a(p^\star - a)} - \frac{2c_1 - p^\star}{(c_1 - a)(a + c_1 - p^\star)}.$$

Notice that $\tau(0) = +\infty$, $\tau(p^\star/2) = 0$, and $\tau'(0) = -\infty$. It suffices to show that there exists at most one solution to $\tau'(a) = 0$ on the interval $\left(0, \frac{p^\star}{2}\right)$.

Simple algebra would verify that $\tau'(a) = 0$ is equivalent to

$$\varphi(\theta) := \mathcal{A}\theta^2 + \mathcal{B}\theta + \mathcal{C} = 0,$$

where $\theta := a(p^\star - a)$ and $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are defined, respectively, as

$$\mathcal{A} := 2c_1 + p^\star(c_2 + c_3 - 1) - 2c_3,$$
$$\mathcal{B} := c_1^2\left[c_2 p^\star + c_3(p^\star - 2)\right] + c_1\left\{2 - p^\star\left[c_2 p^\star + c_3(p^\star - 2) + 2\right]\right\} - p^\star(1 - c_2)(1 - p^\star),$$
$$\mathcal{C} := c_1(c_1 - p^\star)c_2(1 - p^\star)p^\star > 0.$$

Notice that $\varphi(\theta)$ is a quadratic function with $\varphi(0) > 0$ and $\theta = a(p^\star - a) \le (p^\star)^2/4 < \frac{1}{4}$. Therefore, to prove the lemma, it suffices to show that $\varphi(1/4) < 0$. Simple algebra would verify that

$$\varphi\left(\frac{1}{4}\right) = \frac{\begin{bmatrix} 4r^2(p^\star - 1)\{[8(p^\star - 1)p^\star - 1]\,d + 4(7 - 5p^\star)p^\star - 6\} \\ -2r\left[8(3d - 4)(p^\star)^3 + 4(19 - 13d)(p^\star)^2 + (26d - 49)p^\star + 3d + 5\right] \\ +(d - 1)(5 - 4p^\star)^2 p^\star + 16r^3(p^\star - 1)^2(2p^\star - 1) \end{bmatrix}}{16d(1 - 2r)^2}.$$

Define the numerator of the fraction on the right-hand side of the above equation by $\mathcal{J}(d)$. Note that $\mathcal{J}(d)$ is linear in d, and thus it suffices to show that $\mathcal{J}(0) < 0$ and $\mathcal{J}(1) < 1$. It can be verified that

$$\mathcal{J}(0) = \underbrace{\left[4r(2r - 3)(p^\star - 1) + 4p^\star - 5\right]}_{\mathcal{K}(p^\star)} \times \underbrace{\left[4(r - 1)(p^\star)^2 + (5 - 6r)p^\star + 2r\right]}_{\mathcal{L}(r)} \tag{18}$$

Note that $\mathcal{K}(p^\star)$ in (18) is linear in $p^\star$, and thus

$$\mathcal{K}(p^\star) \le \max\{\mathcal{K}(0), \mathcal{K}(1)\} = \max\{4(3 - 2r)r - 5, -1\} \le \max\{-0.5, -1\} < 0,$$

where the second inequality follows from $(3 - 2r)r \le 9/8$. Similarly, $\mathcal{L}(r)$ in (18) is linear in r, which implies that

$$\mathcal{L}(r) > \min\{\mathcal{L}(0), \mathcal{L}(1)\} = \min\left\{p^\star + 4p^\star(1 - p^\star), 2 - p^\star\right\} > 0.$$

Therefore, $\mathcal{J}(0) = \mathcal{K}(p^\star) \times \mathcal{L}(r) < 0$, and it remains to show that $\mathcal{J}(1) < 0$, which is equivalent to

$$\mathcal{V}(p^\star, r) := 2(1 - 2p^\star)(1 - p^\star)\left[-4(1 - p^\star)r^2 + (7 - 6p^\star)r\right] + p^\star\left[8(p^\star - 3)p^\star + 23\right] - 8 < 0.$$

Note that $\mathcal{V}(p^\star, r)$ is quadratic in r. It is straightforward to verify that for all $0 < p^\star < 1$, we have that

$$\mathcal{V}(p^\star, 0) = p^\star\left[8(p^\star - 3)p^\star + 23\right] - 8 < 0,$$

and

$$\mathcal{V}(p^\star, 1) = p^\star - 2 < 0.$$

If $p^\star \ge 1/2$, then $\mathcal{V}(p^\star, r)$ is convex in r. It follows immediately that

$$\mathcal{V}(p^\star, r) \le \max\left\{\mathcal{V}(p^\star, 0), \mathcal{V}(p^\star, 1)\right\} < 0, \forall r \in (0, 1].$$

If $p^\star < 1/2$, then $\mathcal{V}(p^\star, r)$ is concave in r and is maximized at $r = (7 - 6p^\star)/[8(1 - p^\star)] < 1$, which implies that

$$\mathcal{V}(p^\star, r) \le \mathcal{V}\left(p^\star, \frac{7 - 6p^\star}{8(1 - p^\star)}\right) = \frac{1}{8}\left\{2p^\star\left[-4(p^\star)^2 + 6p^\star + 1\right] - 15\right\} < 0, \forall\, 0 < p^\star < 1.$$

Therefore, we have that $\mathcal{J}(1) < 1$. This concludes the proof. ■

Now we can prove Theorem 3. Fix $r, d \in (0, 1]$. We first show that all active firms must win with equal probability. It suffices to prove that

$$\frac{g(a) + g(b)}{2} \ge \min\left\{g\left(\frac{a + b}{2}\right), \frac{g(0) + g(a + b)}{2}\right\}, \quad \forall a, b \ge 0, a + b < 1. \tag{19}$$

We consider the following two cases depending on $d+r$ relative to one.

Case I: $d+r \leq 1$. By Lemma 5, $g(\cdot)$ is convex. Therefore, we have that

$$\frac{g(a)+g(b)}{2} \geq g\left(\frac{a+b}{2}\right) \geq \min\left\{g\left(\frac{a+b}{2}\right), \frac{g(0)+g(a+b)}{2}\right\}, \forall a, b \geq 0, a+b < 1.$$

Case II: $d+r > 1$. Suppose, to the contrary, that there exist (a_0, b_0), with $b_0 > a_0 > 0$ and $a_0 + b_0 < 1$, such that

$$\frac{g(a_0)+g(b_0)}{2} < g\left(\frac{a_0+b_0}{2}\right), \tag{20}$$

and

$$\frac{g(a_0)+g(b_0)}{2} < \frac{g(0)+g(a_0+b_0)}{2}. \tag{21}$$

First, note that it cannot be the case that $b_0 > a_0 \geq \tilde{p}$. Otherwise, it follows from Lemma 5 that $g(\cdot)$ is strictly convex on $[a_0, b_0]$, and thus

$$\frac{g(a_0)+g(b_0)}{2} > g\left(\frac{a_0+b_0}{2}\right),$$

which contradicts (20). Next, note that it cannot be the case that $\tilde{p} \geq b_0 > a_0$. Otherwise, it follows immediately from Lemma 7 and $g(0)=0$ that

$$\frac{g(a_0)+g(b_0)}{2} \geq \frac{g(a_0+b_0)}{2} = \frac{g(0)+g(a_0+b_0)}{2},$$

which is a contradiction of (21). Therefore, we must have that $b_0 > \tilde{p} > a_0$. Lemma 8, together with (20), implies that $b_0 < \hat{p}(a_0)$. Next, define

$$\delta(a,b) := \frac{g(0)+g(a+b)}{2} - \frac{g(a)+g(b)}{2}, \quad \text{for } (a,b) \in [0,\tilde{p}] \times [\tilde{p}, 1).$$

It follows immediately from (21) that $\delta(a_0, b_0) > 0$. Moreover, we have that

$$\frac{\partial \delta}{\partial b} = \frac{g'(a+b) - g'(b)}{2} > 0,$$

where the strict inequality follows from Lemma 5 and $b_0 > \tilde{p}$. Therefore, we have that $\delta(a_0, \hat{p}(a_0)) > \delta(a_0, b_0) > 0$, which is equivalent to

$$\frac{g(a_0)+g(\hat{p}(a_0))}{2} < \frac{g(0)+g(a_0+\hat{p}(a_0))}{2} = \frac{g\left(a_0+\hat{p}(a_0)\right)}{2}. \tag{22}$$

Meanwhile, according to the definition of $\hat{p}(\cdot)$ in Lemma 8, we can obtain that

$$\frac{g(a_0)+g(\hat{p}(a_0))}{2} = g\left(\frac{a_0+\hat{p}(a_0)}{2}\right). \tag{23}$$

The above equation, together with Lemma 9, implies instantly that

$$a_0 + \hat{p}(a_0) \leq p^\star,$$

which in turn implies that

$$\frac{g\left(a_0+\hat{p}(a_0)\right)}{2} \leq g\left(\frac{a_0+\hat{p}(a_0)}{2}\right), \tag{24}$$

by Lemma 6. Combining (23) and (24), we can obtain that

$$\frac{g(a_0)+g(\hat{p}(a_0))}{2} = g\left(\frac{a_0+\hat{p}(a_0)}{2}\right) \geq \frac{g\left(a_0+\hat{p}(a_0)\right)}{2}, \tag{25}$$

which contradicts (22). Therefore, there exists no (a_0, b_0), with $b_0 > a_0 > 0$ and $a_0 + b_0 < 1$, to satisfy (20) and (21) simultaneously.

From the above analysis, we see that (19) holds for all $(r, d) \in (0,1] \times (0,1]$. Therefore, all active firms must win with equal probability, and it remains to pin down the number of active firms in the optimal contest, which we denote by $\kappa^\dagger$. It is evident that $\kappa^\dagger$ solves the following optimization problem:

$$\min_{\kappa' \in \{2,\dots,n\}} \left(\frac{1}{\kappa'}\right)^{\frac{1-d-r}{d}} \Big/ \left(1-\frac{1}{\kappa'}\right)^{\frac{r}{d}}.$$

Simple algebra would verify that

$\kappa^\dagger = n$, if $d \in (0, 1-r]$,

and

$$\kappa^\dagger \in \left\{\min\left\{n, \left\lfloor 1+\frac{r}{r+d-1}\right\rfloor\right\}, \min\left\{n, \left\lfloor 2+\frac{r}{r+d-1}\right\rfloor\right\}\right\}, \text{ if } d \in (1-r, 1].$$

This completes the proof. ■

References

Baye, Michael R., Hoppe, Heidrun C., 2003. The strategic equivalence of rent-seeking, innovation, and patent-race games. Games Econom. Behav. 44 (2), 217–226.

Chan, William, Courty, Pascal, Hao, Li, 2008. Suspense: Dynamic incentives in sports contests. Econom. J. 119 (534), 24–46.

Cornes, Richard, Hartley, Roger, 2005. Asymmetric contests with general technologies. Econom. Theory 26 (4), 923–946.

Dasgupta, Ani, Nti, Kofi O., 1998. Designing an optimal contest. Eur. J. Political Econ. 14 (4), 587–603.

Ely, Jeffrey, Frankel, Alexander, Kamenica, Emir, 2015. Suspense and surprise. J. Political Economy 123 (1), 215–260.

Fang, Dawei, Noe, Thomas, 2019. Less Competition, More Meritocracy? Working Paper.

Fort, Rodney, Quirk, James, 1995. Cross-subsidization, incentives, and outcomes in professional team sports leagues. J. Econ. Lit. 33 (3), 1265–1299.

Franke, Jörg, Kanzow, Christian, Leininger, Wolfgang, Schwartz, Alexandra, 2013. Effort maximization in asymmetric contest games with heterogeneous contestants. Econom. Theory 52 (2), 589–630.

Fu, Qiang, Lu, Jingfeng, 2012. Micro foundations of multi-prize lottery contests: A perspective of noisy performance ranking. Soc. Choice Welf. 38 (3), 497–517.

Fu, Qiang, Lu, Jingfeng, Lu, Yuanzhu, 2012. Incentivizing R&D: Prize or subsidies? Int. J. Ind. Organ. 30 (1), 67–79.

Fu, Qiang, Wu, Zenan, 2020. On the optimal design of biased contests. Theor. Econ. 15 (4), 1435–1470.

Fullerton, Richard L., McAfee, R. Preston, 1999. Auctioning entry into tournaments. J. Political Economy 107 (3), 573–605.

Hvide, Hans K., Kristiansen, Eirik G., 2003. Risk taking in selection contests. Games Econom. Behav. 42 (1), 172–179.

Kirkegaard, René, 2020. Microfounded Contest Design. Working Paper.

Lichtenberg, Frank R., 1990. US government subsidies to private military R&D investment: The defense department's independent R&D policy. Def. Peace Econ. 1 (2), 149–158.

Loury, Glenn C., 1979. Market structure and innovation. Q. J. Econ. 93 (3), 395–410.

Meyer, Margaret A., 1991. Learning from coarse information: Biased contests and career profiles. Rev. Econom. Stud. 58 (1), 15–41.

Runkel, Marco, 2006. Optimal contest design, closeness and the contest success function. Public Choice 129, 217–231.

Ryvkin, Dmitry, Ortmann, Andreas, 2008. The predictive power of three prominent tournament formats. Manage. Sci. 54 (3), 492–504.

Stein, William E., 2002. Asymmetric rent-seeking with more than two contestants. Public Choice 113, 325–336.

Szidarovszky, Ferenc, Okuguchi, Koji, 1997. On the existence and uniqueness of pure Nash equilibrium in rent-seeking games. Games Econom. Behav. 18 (1), 135–140.

Szymanski, Stefan, 2003. The economic design of sporting contests. J. Econ. Lit. 41 (4), 1137–1187.