Multi-Prize Contests with Risk-Averse Players

ONLINE APPENDIX

(Not Intended for Publication)

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In this online appendix, we collect the materials omitted from the the main text of the paper.[1] In Appendix A, we provide proofs of corollaries; In Appendix B, we present additional results under the assumption that contestants exhibit CARA preferences.

A Proof of Corollaries

Proof of Corollary 1

Proof. Exploiting the CARA functional form (i.e., $u(c) = 1 - \exp(-\alpha c)$, with $\alpha > 0$) and the first-order condition (4), we can solve for the unique symmetric equilibrium effort level as follows:

$$e = r \times \frac{\sum_{m=1}^{N} \left[\mu_m \times u(w + V_m - e)\right]}{\sum_{m=1}^{N} u'(w + V_m - e)} = -\frac{r}{\alpha} \times \frac{\sum_{m=1}^{N} \left[\mu_m \times \exp(-\alpha V_m)\right]}{\sum_{m=1}^{N} \exp(-\alpha V_m)}.$$

The above observation, together with Theorem 1, implies instantly that exists a unique symmetric pure-strategy equilibrium of the contest game for all $r \leq 1$. This concludes the proof. ■

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[1] This note is not self-contained; it is the online appendix of the paper "Multi-Prize Contests with Risk-Averse Players."

Proof of Corollary 2

Proof. Note that

$$\frac{d\frac{u'\left(w-\frac{V}{N}\right)}{u'(w+V)}}{dw} = \frac{u''\left(w-\frac{V}{N}\right)u'\left(w+V\right) - u'\left(w-\frac{V}{N}\right)u''\left(w+V\right)}{\left[u'\left(w+V\right)\right]^2}$$
$$= \left[\frac{u''\left(w-\frac{V}{N}\right)}{u'\left(w-\frac{V}{N}\right)} - \frac{u''\left(w+V\right)}{u'\left(w+V\right)}\right] \times \frac{u'\left(w-\frac{V}{N}\right)}{u'\left(w+V\right)} < 0,$$

where the strict inequality follows from the assumption of DARA preferences. Next, note that $\lim_{w\nearrow\infty} u'\left(w-\frac{V}{N}\right)/u'\left(w+V\right) = 1$ by assumption. Therefore, there exists a threshold $\bar{w}$ such that

$$\frac{u'\left(w-\frac{V}{N}\right)}{u'\left(w+V\right)} \leq \frac{r\mu_1 + \frac{V}{N}\tau}{r\mu_2 + \frac{V}{N}\tau}, \text{ for } w > \bar{w},$$

and it is optimal for an effort-maximizing contest designer to award a single prize by Proposition 2. This concludes the proof. ∎

Proof of Corollary 3

Proof. Note that the equilibrium effort level for any prize allocation must be less than $\frac{V}{N}$. Therefore, $e_s < \frac{V}{N}$. Moreover, we have that

$$\frac{u'(w-e_s)}{u'(w+V-e_s)} \geq \frac{u'(w)}{u'\left(w+\frac{N-1}{N}V\right)},$$

and

$$\frac{\mu_1}{\mu_2} = \frac{1-\frac{1}{N}}{1-\frac{1}{N}-\frac{1}{N-1}}.$$

First, note that $\frac{u'(w)}{u'\left(w+\frac{N-1}{N}V\right)}$ is strictly increasing in V and is approaching $u'(w)/u'(\infty)$ as $V \nearrow \infty$. Moreover, the ratio $\frac{\mu_1}{\mu_2}$ is independent of V. Therefore, if $u'(\infty) < \frac{\mu_2}{\mu_1 u'(w)}$, then $\frac{u'(w-e_s)}{u'(w+V-e_s)} > \frac{\mu_1}{\mu_2}$ holds for sufficiently large V.

Second, note that $\frac{u'(w)}{u'\left(w+\frac{N-1}{N}V\right)}$ is strictly increasing in N and is approaching $\frac{u'(w)}{u'(w+V)} > 1$ as $N \nearrow \infty$; and $\frac{1-\frac{1}{N}}{1-\frac{1}{N}-\frac{1}{N-1}}$ is strictly decreasing in N and is approaching 1 as $N \nearrow \infty$. Therefore, $\frac{u'(w-e_s)}{u'(w+V-e_s)} > \frac{\mu_1}{\mu_2}$ as N becomes sufficiently large. This completes the proof. ∎

Proof of Corollary 4

Proof. Note that $u''' = 0$ if the utility function is quadratic. The proof for the optimality of multiple prizes follows immediately from Proposition 3, and it remains to show that it is optimal for the contest designer to award a single prize if $\frac{u'(w-e_s)}{u'(w+V-e_s)} < \frac{\mu_1}{\mu_2}$. It is useful to prove an intermediate result.

Lemma A1 *Suppose that $u(\cdot)$ is strictly increasing, concave, twice continuously differentiable, and exhibits increasing absolute risk aversion (IARA, henceforth). Then*

$$\frac{u'(c_1)}{u'(c_1+\Delta)} < \frac{u'(c_2)}{u'(c_2+\Delta)},\ \textit{for all}\ \Delta > 0\ \textit{and}\ c_1 < c_2.$$

Proof. It suffices to show that $d\left[\frac{u'(c)}{u'(c+\Delta)}\right]/dc > 0$. Note that

$$\begin{aligned}\frac{d}{dc}\left[\frac{u'(c)}{u'(c+\Delta)}\right] &= \frac{u''(c)u'(c+\Delta) - u'(c)u''(c+\Delta)}{\left[u'(c+\Delta)\right]^2} \\ &= \frac{u'(c)}{u'(c+\Delta)} \times \left\{\left[-\frac{u''(c+\Delta)}{u'(c+\Delta)}\right] - \left[-\frac{u''(c)}{u'(c)}\right]\right\} > 0,\end{aligned}$$

where the strict inequality follows directly from the IARA assumption on the utility function. This completes the proof. ■

We are now ready to prove Corollary 4. Suppose to the contrary that

$$\frac{u'(w-e_s)}{u'(w+V-e_s)} < \frac{\mu_1}{\mu_2}, \tag{A1}$$

and awarding multiple prizes is optimal. We follow the notation in the proof of Proposition 2 and denote the optimal prize structure and the corresponding equilibrium effort by $\boldsymbol{V}^* \equiv (V_1^*, \ldots, V_N^*)$ and e^*, respectively. Because awarding multiple prizes is optimal by assumption, it follows immediately from Proposition 1 that $V_1^* > V_2^* > 0$ and $e^* \geq e_s$. Consider an alternative prize allocation that decreases V_2^* by a small amount $\epsilon > 0$, and increases V_1^* by the same amount. Next, we show this alternative allocation generates more effort than $\boldsymbol{V}^*$. Applying the same argument as in the proof of Proposition 2, it suffices to show that

$$\begin{aligned}\zeta'(0) = r \times &\left[\mu_1 u'(w+V_1^*-e^*) - \mu_2 u'(w+V_2^*-e^*)\right] \\ &- e^* \times \left[u''(w+V_1^*-e^*) - u''(w+V_2^*-e^*)\right] > 0.\end{aligned}$$

Note that $u''(c)$ is a constant due to the assumption of quadratic utility function. Therefore,

$\zeta'(0) > 0$ is equivalent to

$$\frac{u'(w + V_2^* - e^*)}{u'(w + V_1^* - e^*)} < \frac{\mu_1}{\mu_2}.$$

Note that

$$\frac{u'(w + V_2^* - e^*)}{u'(w + V_1^* - e^*)} < \frac{u'(w - e^*)}{u'(w + V - e^*)} \leq \frac{u'(w - e_s)}{u'(w + V - e_s)} < \frac{\mu_1}{\mu_2},$$

where the first inequality follows from $0 < V_2^* < V_1^* < V$ and the strict concavity of $u(\cdot)$; the second inequality follows from $e^* \geq e_s$, the fact that a quadratic utility function exhibits IARA, and Lemma A1; and the last inequality follows directly from (A1). This concludes the proof. ■

Proof of Corollary 5

Proof. By the same argument as in the proof of Proposition 3, it suffices to show that

$$G'(0) = r \times \left[-\mu_1 u'(w + V - e_s) + \mu_2 u'(w - e_s)\right] + e_s \times \left[u''(w + V - e_s) - u''(w - e_s)\right] > 0.$$

The first-order condition (4), together with the CARA functional form of utility $u(\cdot)$, implies that

$$\begin{aligned} e_s &= \frac{\sum_{m=1}^{N} \left[\mu_m \times u(w + V_m - e_s)\right]}{\sum_{m=1}^{N} u'(w + V_m - e_s)} r \\ &= -\frac{r}{\alpha} \times \frac{\sum_{m=1}^{N} \left[\mu_m \times \exp(-\alpha V_m)\right]}{\sum_{m=1}^{N} \exp(-\alpha V_m)} \\ &= \frac{r\mu_1}{\alpha} \times \frac{1 - \exp(-\alpha V)}{N - 1 + \exp(-\alpha V)}, \end{aligned}$$

where the last equality follows from the postulated $\boldsymbol{V}_s \equiv (V, 0, \ldots, 0)$. Note that $u''(c) = -\alpha \cdot u'(c)$, and thus $G'(0)$ can be simplified as

$$G'(0) = -(r\mu_1 + \alpha e_s) u'(w + V - e_s) + (r\mu_2 + \alpha e_s) u'(w - e_s).$$

Therefore, $G'(0) > 0$ is equivalent to

$$\frac{u'(w - e_s)}{u'(w + V - e_s)} > \frac{r\mu_1 + \alpha e_s}{r\mu_2 + \alpha e_s} = \frac{N\mu_1}{\mu_1 \left[1 - \exp(-\alpha V)\right] + \mu_2 \left[N - 1 + \exp(-\alpha V)\right]}.$$

This completes the proof. ■

B Additional Results with CARA Preferences

Next, we provide more results with CARA preferences.

Proposition A1 *Suppose that contestants' utility exhibits CARA of $\alpha > 0$ and $N \geq 3$. Then the effort-maximizing contest sets $V_N^* = 0$.*

Proof. Denote the optimal prize allocation by $\boldsymbol{V}^* := (V_1^*, \ldots, V_N^*)$ and the equilibrium effort level by e^*. Suppose that to the contrary that $V_N^* > 0$. Consider an alternative prize schedule $\boldsymbol{V}^\star := (V_1^\star, \ldots, V_N^\star)$, where $V_1^\star = V_1^* + (N-1)N_N^*$ and $V_m^\star = V_m^* - V_N^*$ for all $m \in \{2, \ldots, N\}$; and denote the corresponding equilibrium effort level by $e^\star$. It suffices to show that $e^\star > e^*$.

For notational convenience, let us define μ_{-1} as

$$\mu_{-1} := \frac{\sum_{m=2}^{N} \left[\mu_m \times \exp(-\alpha V_m^*)\right]}{\sum_{m=2}^{N} \exp(-\alpha V_m^*)}.$$

It is straightforward to verify that $\mu_1 > \mu_{-1}$. Exploiting the CARA functional form and the first-order condition (4), we can solve for e^* as

$$\begin{aligned} e^* &= -\frac{r}{\alpha} \times \frac{\sum_{m=1}^{N} \left[\mu_m \times \exp(-\alpha V_m^*)\right]}{\sum_{m=1}^{N} \exp(-\alpha V_m^*)} \\ &= -\frac{r}{\alpha} \times \left[(\mu_1 - \mu_{-1}) \times \frac{\exp(-\alpha V_1^*)}{\sum_{m=1}^{N} \exp(-\alpha V_m^*)} + \mu_{-1}\right]. \end{aligned} \tag{A2}$$

Similarly, $e^\star$ can be derived as

$$\begin{aligned} e^\star &= -\frac{r}{\alpha} \times \frac{\sum_{m=1}^{N} \left[\mu_m \times \exp(-\alpha V_m^\star)\right]}{\sum_{m=1}^{N} \exp(-\alpha V_m^\star)} \\ &= -\frac{r}{\alpha} \times \left[(\mu_1 - \mu_{-1}) \times \frac{\exp(-\alpha(V_1^* + NV_N^*))}{\exp(-\alpha(V_1^* + NV_N^*)) + \sum_{m=2}^{N} \exp(-\alpha V_m^*)} + \mu_{-1}\right]. \end{aligned} \tag{A3}$$

It follows from $V_N^* > 0$ and $\alpha > 0$ that $\exp(-\alpha V_1^*) > \exp(-\alpha(V_1^* + NV_N^*))$, which in turn implies that

$$\frac{\exp(-\alpha V_1^*)}{\sum_{m=1}^{N} \exp(-\alpha V_m^*)} > \frac{\exp(-\alpha(V_1^* + NV_N^*))}{\exp(-\alpha(V_1^* + NV_N^*)) + \sum_{m=2}^{N} \exp(-\alpha V_m^*)}. \tag{A4}$$

Combining (A2), (A3), and (A4), we can obtain that $e^\star > e^*$. This concludes the proof. ■

Proposition A2 *Suppose that contestants' utility exhibits CARA of $\alpha > 0$ and $N = 3$. The effort-maximizing contest sets multiple prizes if $\exp(\alpha V) > 5/2$ and a single prize if $\exp(\alpha V) < 5/2$.*

Proof. Setting $V_3 > 0$ is always suboptimal by Proposition A1. Therefore, the optimal prize schedule must take the form of $\boldsymbol{V} = (V - V_2, V_2, 0)$, where $V_2 \in [0, V/2]$. The condition for the optimality of multiple prizes established in Corollary 5 is $\frac{u'(w-e_s)}{u'(w+V-e_s)} > \frac{3\mu_1}{\mu_1[1-\exp(-\alpha V)]+\mu_2[2+\exp(-\alpha V)]}$ for the case of $N = 3$. Note that $(\mu_1, \mu_2, \mu_3) = (2/3, 1/6, -5/6)$ with $N = 3$, and the condition can be further simplified as $\exp(\alpha V) > 5/2$. Therefore, it remains to show that awarding a single prize is optimal if $\exp(\alpha V) < 5/2$.

With slight abuse of notation, denote by $e(V_2)$ the equilibrium effort level under a prize schedule $\boldsymbol{V} = (V - V_2, V_2, 0)$. Exploiting the CARA functional form and the first-order condition (4), we can obtain that

$$e(V_2) = -\frac{r}{6\alpha} \times \frac{4\exp(-\alpha(V - V_2)) + \exp(-\alpha V_2) - 5}{\exp(-\alpha(V - V_2)) + \exp(-\alpha V_2) + 1}.$$

Taking derivative of $e(V_2)$ with respect to V_2 yields that

$$\begin{aligned} e'(V_2) &= \frac{r}{2} \times \frac{2\exp(-\alpha V_2) - 3\exp(-\alpha(V - V_2)) - 2\exp(-\alpha V)}{\left[\exp(-\alpha(V - V_2)) + \exp(-\alpha V_2) + 1\right]^2} \\ &\leq \frac{r}{2} \times \frac{2 - 5\exp(-\alpha V)}{\left[\exp(-\alpha(V - V_2)) + \exp(-\alpha V_2) + 1\right]^2} < 0, \end{aligned}$$

where the first inequality follows from $\exp(-\alpha V_2) \leq 1$ and $\exp(-\alpha(V - V_2) \geq \exp(-\alpha V)$, and the second inequality follows from $\exp(\alpha V) < 5/2$. Therefore, $e(V_2)$ is strictly decreasing in V_2 for $V_2 \in [0, V/2]$ given that $\exp(\alpha V) < 5/2$, and thus total effort is maximized at $V_2 = 0$. This completes the proof. ■