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Expectations-based loss aversion in contests ☆

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ABSTRACT

This paper studies a multi-player lottery contest in which heterogeneous contestants exhibit reference-dependent loss aversion à la Kőszegi and Rabin (2006, 2007). We verify the existence and uniqueness of pure-strategy choice-acclimating personal Nash equilibrium (CPNE) under moderate loss aversion and fully characterize the equilibrium. The equilibrium sharply contrasts with that in the two-player risk-neutral symmetric case. Loss aversion can lead contestants' individual efforts to change nonmonotonically, while the total effort strictly decreases. Further, it always leads to a more elitist distributional outcome, in the sense that a smaller set of contestants remain active in the competition and stronger contestants' equilibrium winning probabilities increase. We demonstrate that loss aversion generates a fundamentally different decision problem than risk aversion and develop a rationale that explains the contrasting predictions from the two frameworks. Finally, our results are robust under the alternative equilibrium concept of preferred personal Nash equilibrium (PPNE).

1. Introduction

Contest-like situations are ubiquitous in the social and economic landscape. A plethora of competitive events exemplify a contest, ranging from electoral competitions to military conflicts, lobbying, college admissions, and sporting events. In all of these scenarios, contenders exert costly efforts to vie for limited prizes, while their input is nonrefundable regardless of win or loss.

A vast literature examines the strategic substance of contest games. The majority of these studies assume risk-neutral contestants. A contest environment nonetheless provides a natural and relevant "laboratory" to examine the implications

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of economic agents' risk attitudes for their competitive behaviors and strategic decisions amid risk and uncertainty. The stochastic nature of a contest causes inherent uncertainty in contestants' payoffs, while their effort choices affect the probabilistic distribution of contest outcomes and the variation in payoffs across states, which thus endogenously determine the level of risk involved in the game. Contestants' risk attitudes fundamentally reshape their competitive incentives and strategic interactions. The resultant equilibrium behavior would presumably deviate from the prediction of the usual risk-neutral benchmark. This paper aims to shed light on the strategic interactions in contests while allowing for richer contestants' preferences.

The economics literature often examines choices amid risk and uncertainty in a risk-averse model, and a handful of studies have incorporated risk aversion in contest settings.[1] We instead consider expectations-based loss-averse contestants à la Kőszegi and Rabin (2006, 2007) in a multi-player noisy contest in which contestants differ in their valuations of the prize.

A typical loss-averse contestant is defined by two behavioral traits: He (i) evaluates gain and loss against a reference point (*reference-dependent preferences*) and (ii) perceives a loss more significantly than a gain of the same magnitude (*loss aversion*). As pointed out by O'Donoghue and Sprenger (2018), the notion of loss aversion plausibly rationalizes a wide variety of economic behaviors and, in particular, those that conflict with the implications of standard expected-utility frameworks with risk-averse agents. Further, the reference points, as proposed by Kőszegi and Rabin (2006, 2007, 2009), are endogenously formed based on contestants' rational expectations about possible outcomes. Increasing evidence in both the field and laboratory supports the nontrivial roles played by expectations in forming reference points.[2] The notion of expectations-based loss aversion, as a compelling alternative to describe risk attitude, lays a foundation for coherent and disciplined analysis of decision problems in a broad array of contexts.

Our setting enables a rich analysis of contests with expectations-based loss-averse players. The heterogeneous contestants could respond to such preferences in fundamentally different ways, since they perceive gain or loss differently due to their disparate expectations about the outcomes and, therefore, different reference points. The heterogeneous responses triggered by loss aversion are further diffused and compounded through contestants' strategic interaction. Notably, a contest game generates distinctively intricate strategic substance. As Dixit (1987) demonstrates, players' best responses are often nonmonotone in contest games: In contrast to Cournot or Bertrand competitions, one's effort choice is a strategic complement to that of his opponent when he is in the lead, while it is a strategic substitute when he is behind. It remains unclear a priori how the equilibrium interactions unfold and depart from those under standard preferences, since this involves both the direct effect of loss aversion on individual contestants and the subsequent strategic spillover.

Kőszegi and Rabin (2007) develop the notion of *choice-acclimating personal equilibrium* (CPE) to depict the consistent behavior of expectations-based loss-averse *individuals*. Under CPE, an agent's action influences future outcomes and, ultimately, his reference point; his action choice internalizes its implications for expectations about future outcomes and the gain or loss evaluated against the expectations-based reference point. Recently, Dato et al. (2017) and Dato et al. (2018) extend this concept to a game-theoretical model. Following Dato et al. (2017) and Dato et al. (2018), we adopt *choice-acclimating personal Nash equilibrium* (CPNE) as the solution concept.

Incentive effects of loss aversion in contests Contestants' effort choices must strike a balance between the material utility derived from the contest and the psychological gain-loss (dis)utility. The disutility caused by loss aversion is proportional to $p_i(1-p_i)$, where p_i is contestant i's probability of winning. The term $p_i(1-p_i)$ measures the uncertainty in outcomes expected by the contestant. Two effects arise in the contest game. Let us interpret them in a simple case of two heterogeneous contestants who differ in their prize valuations.

First, a *(direct) uncertainty-reducing effect* comes into play. Loss-averse contestants are compelled to reduce the psychological disutility. The weaker contestant tends to cut back on his effort because this further diminishes his winning odds. Intuitively, he is pessimistic about his chances of winning, so he lowers his ex ante input to limit the pain in the event of losing the competition. In contrast, the stronger contestant would expect a relatively optimistic outcome, and tends to increase effort to prevent losing unexpectedly: A higher effort increases p_i and decreases uncertainty.

Second, an *(indirect) competition effect* ensues: When the uncertainty-reducing effect causes each individual contestant to adjust his effort choice, his opponent must respond strategically. Recall the aforementioned nonmonotone best responses in contests. When the uncertainty-reducing effect encourages the favorite to step up his effort, the underdog is further discouraged, since he expects smaller odds to win: Both the (indirect) competition effect and the (direct) uncertainty-reducing effect lead the underdog to concede further. In contrast, when the underdog cuts back on his effort, the favorite is tempted to reduce his effort in response, since a less competitive opponent allows the latter to slack off without suffering lower winning odds: The two effects oppose each other, and the overall effect on the favorite's equilibrium effort is ambiguous and depends on the degree of player heterogeneity.

It is important to stress that both effects are *nonmonotone* in nature and depend on a contestant's relative standing in the competition. The former is driven by the reference-dependent preferences: Heterogeneous contestants form different

[1] See Skaperdas and Gan (1995); Konrad and Schlesinger (1997); Cornes and Hartley (2003b, 2012b); Schroyen and Treich (2016); and Fu et al. (2021a).

[2] Notable contributions include Abeler et al. (2011); Crawford and Meng (2011); Gill and Prowse (2012); Banerji and Gupta (2014); Song (2016); Berger et al. (2018); Goette et al. (2019); and Dreyfuss et al. (2021); among many others.

expectations; gain or loss is thus perceived disparately, which compels a favorite to take a different action to reduce uncertainty vis-à-vis an underdog. The latter effect is caused by the nature of strategic interactions in contests. The interaction of the two effects triggers intricate responses to loss aversion from heterogeneous contestants, which are demonstrated by our results.

Snapshot of results We first verify that a unique CPNE in pure strategy exists for moderate levels of loss aversion.[3] Gill and Prowse (2012) and Dato et al. (2018) show that the CPNE in a two-player symmetric contest coincides with the Nash equilibrium (NE) under standard preferences. In contrast, we demonstrate that the CPNE departs from the NE whenever (i) they are heterogeneous, and/or (ii) the number of contestants exceeds two. For instance, when the contest involves three or more symmetric contestants, they uniformly reduce their efforts when loss aversion is present.

The tension between the uncertainty-reducing effect and the competition effect determines the equilibrium. In the aforementioned two-player asymmetric contest, both effects lead the weaker contestant to decrease his effort. In contrast, the two forces conflict with each other for the stronger, and he decreases his effort if and only if the contest is sufficiently lopsided.

More intricate responses arise when three or more heterogeneous contestants are involved. Bottom contestants are discouraged: They reduce their efforts and may even drop out of the competition by placing a zero bid. Subtler effects, however, loom large for those in the upper bracket: They may either increase or decrease their efforts, and their responses can be nonmonotone, in the sense that the top contestant slacks off, while those in the middle step up their bids. Our analysis provides a complete account of three possible patterns that may emerge in the equilibrium. We elaborate on the logic that underpins the results based on the tension between the (nonmonotone) uncertainty-reducing and competition effects in Section 3.2.

Despite the mixed responses in individual equilibrium efforts, we obtain unambiguous observations about the effect of loss aversion on aggregate incentive and the equilibrium winning probability profile. Overall effort always drops in the contest, regardless of the diverging responses of individual contestants. A more *elitist* redistribution pattern may arise: Loss aversion leads to a smaller set of active contenders, and stronger contestants always end up with higher winning odds.

We further examine two alternative settings. First, we consider a model of risk-averse contestants and demonstrate that its prediction differs from that of our expectations-based loss-aversion setting. We develop a rationale that sheds light on the contrasting roles played by risk aversion vis-à-vis loss aversion in shaping contestants' choices. Second, we show that our main results obtained under CPNE are largely robust when an alternative equilibrium concept—i.e., the preferred personal Nash equilibrium (PPNE)—is adopted.

Related literature The notion of expectations-based loss aversion à la Kőszegi and Rabin (2006, 2007, 2009) has been applied to a broad spectrum of decision problems, such as household insurance choice (Barseghyan et al., 2013); household consumption choice (Kőszegi and Rabin, 2009; Pagel, 2017); firms' marketing and pricing strategies (Herweg and Mierendorff, 2013; Heidhues and Kőszegi, 2014; Karle and Peitz, 2014; Rosato, 2016; Carbajal et al., 2016; Hahn et al., 2018); and optimal wage schemes (Herweg et al., 2010). The numerous studies along this line have mainly focused on stand-alone decision making.

Our paper contributes to the burgeoning literature on the strategic interactions between loss-averse players that sheds light on how their strategic choices and interactions are governed by such preferences. Gill and Stone (2010) and Dato et al. (2018) pioneer the study of contests/tournaments with the presence of expectations-based loss aversion.[4] Both studies consider two-player simultaneous-move rank-order tournament models and primarily focus on the equilibrium fundamentals in the games.[5] Our study differs from these in both setting and focus. We consider a multi-player Tullock contest with heterogeneous contestants and provide a thorough account of the impact of loss aversion on contestants' incentives and equilibrium outcomes.

A handful of studies incorporate expectations-based loss aversion in auction models. Lange and Ratan (2010) show that predictions of bidders' behavior largely depend on whether the auctioned items and money are consumed along the same dimension. Eisenhuth and Grunewald (2018) compare first-price auctions with all-pay auctions, and show that the revenue ranking also depends on how individuals evaluate gain and loss. Rosato and Tymula (2019) provide experimental evidence for the difference in bidding behavior in real-item auctions vis-à-vis induced-value auctions. Balzer and Rosato (2021) study common-value auctions, while Rosato (2019) analyzes sequential auctions. Mermer (2017) investigates optimal revenue-maximizing prize allocation in an all-pay auction model, and shows that a contest designer may split her prize purse into several uniform prizes when contestants are loss averse. Eisenhuth (2019) studies a revenue-maximizing mechanism and shows that the optimal auction is an all-pay auction with a minimum bid when gain and loss are evaluated in separable dimensions. Auction studies typically assume incomplete information and ex ante symmetric bidders, which yield equilibrium bidding strategies as functions of bidders' private types. In contrast, we consider a complete-information Tullock contest

[3] Gill and Stone (2010) and Dato et al. (2018) show that CPNE may cease to exist when contestants are highly loss averse. We focus on the case of moderate loss aversion in the main text; analysis of the contest game under strong loss aversion is provided in Online Appendix A.

[4] Relatedly, Gill and Stone (2015) and Daido and Murooka (2016) adopt reference-dependent preferences in models of team production.

[5] Gill and Prowse (2012) present a theoretical model in which contestants move sequentially. Rosato (2017) considers a sequential negotiation model that allows for a loss-averse buyer.

model. A pure-strategy equilibrium exists, which allows us to model ex ante asymmetric competition and explicitly explore the implications of loss aversion for strategic interactions between heterogeneous players.

Our paper also contributes to the thin literature on contests with behavioral elements. Anderson et al. (1998) allow for boundedly rational bidders in all-pay auctions. Baharad and Nitzan (2008) provide a rationale for rent under-dissipation based on probability distortion. Cornes and Hartley (2012a); Müller and Schotter (2010); and Chen et al. (2017) introduce non-expectations-based loss aversion in contest models. Keskin (2018) studies a multi-player rent-seeking contest in which contestants have cumulative prospect theory (CPT) preferences. He characterizes the endogenous CPT equilibrium (Keskin, 2016) and compares the equilibrium prediction with those from expected-utility-theory models.

The remainder of the paper is organized as follows. Section 2 sets up the model and presents a preliminary analysis that establishes the existence and uniqueness of CPNE in a generalized lottery contest model under moderate loss aversion. Section 3 characterizes the equilibrium under a more specific contest technology and examines the impact of expectations-based loss aversion on contestants' incentives and equilibrium outcomes. Section 4 considers two extensions, and Section 5 concludes. All proofs are collected in the Appendix A, and additional results are relegated to an online appendix.

2. Model and preliminaries

There are $N \geq 2$ contestants competing for a prize. The prize bears a value v_i for each contestant $i \in \mathcal{N} \equiv \{1, \ldots, N\}$, which is common knowledge. Without loss of generality, we assume $v_1 \geq \cdots \geq v_N > 0$.

2.1. Winner-selection mechanism

Contestants simultaneously exert nonnegative efforts to compete for the prize. We consider a generalized lottery contest, with its winner selected through a ratio-form contest success function (CSF): For a given effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_N)$, a contestant i wins with a probability

$$p_i(\boldsymbol{x}) = \begin{cases} \frac{f_i(x_i)}{\sum_{j=1}^N f_j(x_j)} & \text{if } \sum_{j=1}^N x_j > 0, \\ \frac{1}{N} & \text{if } \sum_{j=1}^N x_j = 0, \end{cases} \tag{1}$$

where the function $f_i(\cdot)$ converts one's effort entry into his effective bid in the lottery and is typically labeled the *impact function* in the contest literature.[6] A Tullock contest provides the most salient special case of this winner-selection mechanism, which assumes an impact function $f_i(x_i) = (x_i)^r$.[7] We impose the following conditions on $\{f_i(\cdot)\}_{i=1}^N$.

Assumption 1. $f_i(\cdot)$ is a twice-differentiable function, with $f_i'(x_i) > 0$, $f_i''(x_i) \leq 0$, and $f_i(0) = 0$.

2.2. Contestants' preferences

Contestants are assumed to be expectations-based loss averse, as in Kőszegi and Rabin (2006). To put this formally, fixing opponents' effort profile $\boldsymbol{x}_{-i} \equiv (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$, a contestant i's expected payoff for exerting x_i when he expects himself to exert effort $\hat{x}_i$, denoted by $U_i(x_i, \hat{x}_i, \boldsymbol{x}_{-i})$, is given by

$$\begin{aligned} U_i(x_i, \hat{x}_i, \boldsymbol{x}_{-i}) = {} & p_i(x_i, \boldsymbol{x}_{-i}) \times \Big\{ v_i + \eta \left[1 - p_i(\hat{x}_i, \boldsymbol{x}_{-i})\right] \times \mu(v_i) \Big\} \\ & + \left[1 - p_i(x_i, \boldsymbol{x}_{-i})\right] \times \Big\{ 0 + \eta p_i(\hat{x}_i, \boldsymbol{x}_{-i}) \times \mu(-v_i) \Big\} - x_i + \eta \mu(\hat{x}_i - x_i), \end{aligned} \tag{2}$$

where the parameter $\eta \geq 0$ is the weight a contestant attaches to his gain-loss utility relative to his material utility; $\mu(\cdot)$ is the universal psychological gain-loss utility and is defined as the following:

$$\mu(c) = \begin{cases} c & \text{if } c \geq 0, \\ \lambda c & \text{if } c < 0. \end{cases}$$

The parameter λ is assumed to exceed one, which captures a contestant's loss aversion in the sense that he is more sensitive to a loss than to a gain of the same magnitude.

By Equation (2), the contestant, when expecting that he will exert an effort $\hat{x}_i$, would perceive a gain of $\mu(\hat{x}_i - x_i) = \hat{x}_i - x_i > 0$ when his effort x_i is below his expectation $\hat{x}_i$, and sense a loss of $\left|\mu(\hat{x}_i - x_i)\right| = \lambda \left|\hat{x}_i - x_i\right|$ otherwise.[8] Furthermore,

[6] Two rationales for the microeconomic underpinning of the popularly adopted CSF (1) are provided in the literature: (i) a noisy-ranking approach adapted from the discrete-choice model (Clark and Riis, 1996; Jia, 2008) and (ii) a research tournament analogy (Loury, 1979; Dasgupta and Stiglitz, 1980; Fullerton and McAfee, 1999; Baye and Hoppe, 2003).

[7] The parameter r indicates the precision of the winner-selection mechanism in the contest. A larger r implies that a higher effort can more effectively increase one's winning odds. In the extreme case of r approaching infinity, the contest boils down to an all-pay auction in which a higher bid guarantees a sure win. Note that Assumption 1 requires $0 < r \leq 1$.

[8] We restrict our attention to pure strategy without loss of generality. It can be shown that a choice-acclimating Nash equilibrium in mixed strategies does not exist by an argument similar to the proof of Proposition 3 in Dato et al. (2018).

he expects to win with probability $p_i(\hat{x}_i, \boldsymbol{x}_{-i})$ and lose with probability $1 - p_i(\hat{x}_i, \boldsymbol{x}_{-i})$. This forms his stochastic reference point along the prize dimension. A contestant compares the realized outcome of the contest with each possible outcome in the reference lottery. In particular, winning the contest feels like a gain of $[1 - p_i(\hat{x}_i, \boldsymbol{x}_{-i})] \times \mu(v_i)$, while losing it generates a loss of $p_i(\hat{x}_i, \boldsymbol{x}_{-i}) \times |\mu(-v_i)|$.

We assume that contestants evaluate prize and effort separately when deriving expression (2). Lange and Ratan (2010) consider first-price and second-price auctions with expectations-based loss-averse bidders. They contend that model predictions differ substantially when bidders evaluate their gain and loss from money and the auction item separately vis-à-vis when they evaluate them jointly based on the net utility of the transaction. In contrast, our results would be immune to such modeling nuances due to the all-pay feature of the contest game whenever contestants play pure strategies.[9]

2.3. Equilibrium concepts

Our analysis primarily focuses on the solution concept of CPNE. The notion of CPE (Kőszegi and Rabin, 2007) requires that reference points be formed through rational expectations, and a loss-averse agent's action choice fully internalizes its impact on his expectations and the gain-loss utility measured against the expectations-based reference point. Dato et al. (2017) and Dato et al. (2018) integrate this notion into the analysis of strategic interaction between loss-averse players and develop the solution concept of CPNE. In our context, a CPNE is formally defined as follows.

Definition 1. (Choice-acclimating personal Nash equilibrium) The effort profile $\boldsymbol{x}^* \equiv (x_1^*, \ldots, x_N^*)$ constitutes a choice-acclimating personal Nash equilibrium (CPNE) in pure strategy if for all $i \in \mathcal{N}$,

$$U_i(x_i^*, x_i^*, \boldsymbol{x}_{-i}^*) \geq U_i(x_i, x_i, \boldsymbol{x}_{-i}^*), \text{ for all } x_i \in [0, \infty).$$

By Definition 1, each contestant's expectation about future outcomes will have fully adapted to his actual strategic choice when the uncertainty is resolved; he then commits to a strategy that maximizes his expected utility given his opponents' strategy profile. In other words, the expectation is choice acclimating. The notion of choice acclimating, according to Kőszegi and Rabin (2007), is more plausible when the action is chosen long before the outcome of the contest is realized, and thus each contestant's belief can eventually be adapted to the effort level he has chosen.[10]

2.4. Equilibrium existence and uniqueness

A CPNE requires $x_i = \hat{x}_i$ for all $i \in \mathcal{N}$. Integrating the condition into expression (2) and carrying out the algebra yield

$$\widehat{U}_i(x_i, \boldsymbol{x}_{-i}) := U_i(x_i, x_i, \boldsymbol{x}_{-i}) = \underbrace{p_i(x_i, \boldsymbol{x}_{-i})v_i - x_i}_{\text{material utility}} \underbrace{-\eta(\lambda - 1)p_i(x_i, \boldsymbol{x}_{-i})[1 - p_i(x_i, \boldsymbol{x}_{-i})]v_i}_{\text{gain-loss utility}}. \tag{3}$$

Obviously, our setup degenerates to a standard contest model if the second term vanishes, i.e., if $\eta(\lambda - 1) = 0$. For notational convenience, let us denote $\eta(\lambda - 1)$ by k. This is the overall weight in the contestant's expected utility attached to the net loss caused by loss aversion (see also Herweg et al., 2010; Dato et al., 2018), and hence can be viewed as a composite measure of the intensity of contestants' reference-dependent loss aversion.

Simple math verifies that a contestant's expected utility $\widehat{U}_i(\cdot)$ is strictly concave in his effort x_i for $k \leq 1/2$.[11] A contestant's effort choice can therefore be pinned down by the prevailing first-order condition. Denote by $BR_i(\boldsymbol{x}_{-i})$ a contestant i's best response, which can be derived as the following:

$$BR_i(\boldsymbol{x}_{-i}) = \begin{cases} 0 & \text{if } \left.\frac{\widehat{U}_i(x_i, \boldsymbol{x}_{-i})}{\partial x_i}\right|_{x_i=0} \leq 0, \\ \text{the unique solution to } \frac{\widehat{U}_i(x_i, \boldsymbol{x}_{-i})}{\partial x_i} = 0 & \text{otherwise.} \end{cases}$$

A CPNE is thus an effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_N)$ with $x_i = BR_i(\boldsymbol{x}_{-i})$ for all $i \in \mathcal{N}$.

[9] In a first-price or second-price auction, a bid incurs a cost if and only if one wins. The evaluation of gain and loss in different outcomes thus depends on whether the auction item and money are consumed in separable dimensions of consumption space. In contrast, a contest requires a nonrecoverable bid, and the effort cost is sunk irrespective of the realized outcome. Therefore, the evaluation is independent of the nuance.

[10] Kőszegi and Rabin (2006, 2007) propose another equilibrium concept, the *(preferred) personal equilibrium*, to depict the scenario in which a player makes his decision shortly before the outcome is realized, which prevents his past expectations from being adapted to his actual action choice, i.e., contestants' expectations are choice unacclimating. Our main results are robust to this alternative equilibrium concept. See Section 4.2 for more discussion.

[11] To see this, note that $\frac{\partial^2 \widehat{U}_i}{\partial x_i^2} = (1 - p_i)p_i v_i \times \left\{(1 - k + 2kp_i)\frac{f_i''(x_i)}{f_i(x_i)} + (-2 + 4k - 6kp_i)p_i\left[\frac{f_i'(x_i)}{f_i(x_i)}\right]^2\right\}$. Moreover, we have $f_i''(x_i) \leq 0$ from Assumption 1; and $-2 + 4k - 6kp_i \leq -6kp_i < 0$ for $k \leq \frac{1}{2}$ and $p_i > 0$. Therefore, $\frac{\partial^2 \widehat{U}_i}{\partial x_i^2} < 0$ for $x_i > 0$ if $k \leq \frac{1}{2}$.

Szidarovszky and Okuguchi (1997); Stein (2002); and Cornes and Hartley (2005) establish the existence and uniqueness of Nash equilibrium in the contest game under standard preferences, which corresponds to the case of $k = 0$ in our setup. We now demonstrate that this result can be retained when contestants are moderately loss averse à la Kőszegi and Rabin (2007).

Proposition 1. *Suppose that Assumption 1 is satisfied and* $k \equiv \eta(\lambda - 1) \in [0, \frac{1}{3}]$. *Then there exists a unique pure-strategy CPNE of the contest game.*

Proposition 1 requires moderate loss aversion. It is well known in the literature that a CPNE may fail to exist when contestants are excessively loss averse.[12] Our analysis mainly focuses on the case of $k \leq 1/3$; the implications of a large k will be discussed in Online Appendix A.

3. Equilibrium analysis

In this section, we characterize the unique CPNE in the contest game and delineate how expectations-based loss-averse contestants' incentive and behavior depart from those of their counterparts with standard preferences. To gain more mileage, we focus on the popularly adopted lottery contest model with a linear impact function (see Stein, 2002, and Franke et al., 2013, among many others). The following assumption is imposed throughout the rest of the section.

Assumption 2. $f_i(x_i) = x_i$ for all $i \in \mathcal{N}$.

In Online Appendix B, we relax the restriction of a linear impact function and show that our main results are robust. Denote by $\boldsymbol{x}^* \equiv (x_1^*, \ldots, x_N^*)$ the equilibrium effort profile in the unique CPNE, which is fully characterized in the next result.

Proposition 2. *Suppose that Assumption 2 is satisfied and* $k \equiv \eta(\lambda - 1) \in \left[0, \frac{1}{3}\right]$. *In the unique CPNE, contestant i's equilibrium effort* x_i^*, *with* $i \in \mathcal{N}$, *is given by*

$$x_i^* = g_i(s) = \begin{cases} 0 & \text{if } (1-k)v_i \leq s, \\ \dfrac{\sqrt{(1-3k)^2 s^2 + 8ks^2\left(1-k-\frac{s}{v_i}\right)} - (1-3k)s}{4k} & \text{otherwise,} \end{cases} \tag{4}$$

where $s > 0$ *is the unique solution to* $\sum_{i=1}^N g_i(s) = s$.

The equilibrium characterization relies on a key property of the contest as an aggregative game: A contestant's expected payoff (3) depends only on his individual output x_i and the total effort $\sum_{j=1}^N x_j$, which enables the powerful tool of backward-reply correspondence (see Selten, 1970; Novshek, 1985; Acemoglu and Jensen, 2013) in equilibrium analysis and, subsequently, share correspondence (Cornes and Hartley, 2005) in contests. In fact, s is the aggregate equilibrium effort of all contestants.

Next, we investigate the impact of reference-dependent preferences on contestants' equilibrium effort choice. For this purpose, we slightly abuse the notation and write x_i^* as $x_i^*(k)$, i.e., a function of the degree of loss aversion k. It can be verified that $x_i^*(k)$ is differentiable almost everywhere.

3.1. Contests with two contestants: $N = 2$

Although the equilibrium effort profile $\boldsymbol{x}^*(k) := \left(x_1^*(k), \ldots, x_N^*(k)\right)$ is fully characterized in Proposition 2, a closed-form solution is unavailable in general because the total effort in equilibrium, s, is implicitly determined by the condition $\sum_{i=1}^N g_i(s) = s$. To provide a lucid account of the impact of reference-dependent preferences, it is useful to first restrict our attention to a two-player case, as in the literature (e.g., Gill and Stone, 2010, 2015; Dato et al., 2017).

Assuming $N = 2$, the equilibrium effort profile $\left(x_1^*(k), x_2^*(k)\right)$ can be solved explicitly as the following:

Corollary 1. *Suppose that Assumption 2 is satisfied,* $k \in [0, \frac{1}{3}]$, *and* $N = 2$. *The equilibrium effort pair* $\left(x_1^*(k), x_2^*(k)\right)$ *is given by*

$$x_1^*(k) = \frac{\theta}{(1+\theta)^2} v_1 - \frac{\theta(1-\theta)}{(1+\theta)^3} k v_1,$$

and

[12] See Dato et al. (2017) and Dato et al. (2018) for detailed discussion of the nonexistence of CPNE.

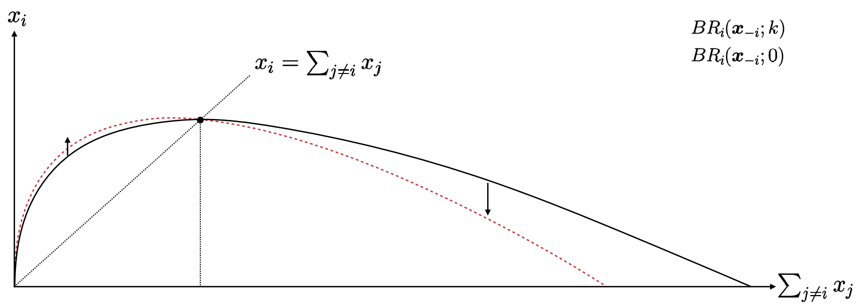


Fig. 1. Best Response Function for Contestant i: $x_i = BR_i(\boldsymbol{x}_{-i}; k)$.

$$x_2^*(k) = \frac{1}{(1+\theta)^2} v_1 - \frac{1-\theta}{(1+\theta)^3} k v_1,$$

where

$$\theta = \frac{1}{2}\left[\left(\frac{v_1}{v_2} - 1\right) \times \frac{1+k}{1-k} + \sqrt{\left(\frac{v_1}{v_2} - 1\right)^2 \times \left(\frac{1+k}{1-k}\right)^2 + \frac{4v_1}{v_2}}\right].$$

A closer look at the equilibrium result leads to the following comparative statics.

Proposition 3. *Suppose that Assumption 2 is satisfied and $N = 2$. The following statements hold:*

(i) *If $v_1 = v_2 =: v$, then $x_1^*(k) = x_2^*(k) = \frac{1}{4} v$ for $k \in [0, \frac{1}{3}]$ and hence $\left.\frac{dx_1^*}{dk}\right|_{k=0} = \left.\frac{dx_2^*}{dk}\right|_{k=0} = 0$.*

(ii) *If $v_1 > v_2$, then $\left.\frac{dx_2^*}{dk}\right|_{k=0} < 0$. Moreover, $\left.\frac{dx_1^*}{dk}\right|_{k=0} > 0$ if and only if $\frac{v_1}{v_2} < 3$.*

Part (i) of Proposition 3 states that when contestants are homogeneous, the unique pure-strategy CPNE is symmetric and identical to the unique Nash equilibrium for contestants with standard preferences. This observation echoes the findings of Gill and Stone (2010, Proposition 2) and Dato et al. (2017, Proposition 1) in alternative contest settings. However, part (ii) of Proposition 3 demonstrates that loss aversion plays a significant role for heterogeneous contestants, which causes the predictions to diverge from those in a standard framework. Loss aversion reduces the weak contestant's equilibrium bid; the strong contestant may either increase or decrease his effort, depending on the degree of heterogeneity between the contestants. When the dispersion of contestants' prize valuations remains moderate, i.e., $v_1/v_2 < 3$, the strong contestant exerts more effort under loss aversion; he nevertheless decreases his effort level when the competition is excessively asymmetric, i.e., $v_1/v_2 > 3$.

Incentive effects of loss aversion: decomposition Recall that when contestants' expectations are choice acclimating, one's utility is given by

$$\widehat{U}_i(x_i, \boldsymbol{x}_{-i}) = \underbrace{p_i(x_i, \boldsymbol{x}_{-i}) v_i - x_i}_{\text{material utility}} \underbrace{- k p_i(x_i, \boldsymbol{x}_{-i})[1 - p_i(x_i, \boldsymbol{x}_{-i})] v_i}_{\text{gain-loss utility}}.$$

The psychological gain-loss utility is proportional to $p_i(x_i, \boldsymbol{x}_{-i})[1 - p_i(x_i, \boldsymbol{x}_{-i})]$, which is a natural measure of the uncertainty regarding the outcome of the contest. A loss-averse contestant dislikes uncertainty and is inclined to take action to reduce it. We now decompose the incentive effect into two sources.

First, a *(direct) uncertainty-reducing effect* is caused by expectations-based loss aversion. Note that the uncertainty measure, $p_i(x_i, \boldsymbol{x}_{-i})[1 - p_i(x_i, \boldsymbol{x}_{-i})]$, strictly increases with $p_i(x_i, \boldsymbol{x}_{-i})$ first and strictly decreases once $p_i(x_i, \boldsymbol{x}_{-i})$ exceeds 1/2. A loss-averse contestant, to reduce uncertainty about the outcome, is tempted to decrease (increase) his effort if his winning probability falls below (exceeds) the threshold 1/2: The underdog is poised to drive down $p_i(x_i, \boldsymbol{x}_{-i})$ toward zero, while the favorite would push it toward one. To put this more intuitively, the underdog expects a less likely win—which compels him to reduce unnecessary efforts—while the favorite steps up his effort to avert an inadvertent loss.

The effect is illustrated with Fig. 1, which plots a contestant's best response in the contest game with and without loss aversion. The presence of loss aversion causes an inward rotation of the best response curve: With $k > 0$, a contestant steps up his bid in his best response to a given $\sum_{j \neq i} x_j$ for $x_i > \sum_{j \neq i} x_j$; he backs off for $x_i < \sum_{j \neq i} x_j$.

The uncertainty-reducing effect triggers an *(indirect) competition* effect that comes into play through the strategic interaction between contestants, since each must adjust his effort choice in response to the change in the effort of his opponent.

As stated previously, opponents' efforts are strategic complements to a contestant i if he is in the lead (i.e., $x_i > \sum_{j \neq i} x_j$), and strategic substitutes otherwise (i.e., $x_i < \sum_{j \neq i} x_j$).

In our context, on the one hand, a more aggressive favorite—due to the uncertainty-reducing effect—further disincentivizes the underdog because of the strategic substitutability; on the other hand, the concession of the underdog allows the favorite to slack off because of the strategic complementarity, since a lower effort may still render him equally likely to win.

Interpretation This rationale sheds immediate light on the knife-edge case of symmetric two-player contests, in which each contestant wins with an equal probability. Define $\widetilde{U}_i(x_i, \boldsymbol{x}_{-i}) := -kp_i(x_i, \boldsymbol{x}_{-i})[1 - p_i(x_i, \boldsymbol{x}_{-i})]v_i$, which is a contestant i's gain-loss utility. It follows immediately that

$$\frac{\partial \widetilde{U}_i(x_i, \boldsymbol{x}_{-i})}{\partial x_i} = -k\left[1 - 2p_i(x_i, \boldsymbol{x}_{-i})\right] \frac{\partial p_i(x_i, \boldsymbol{x}_{-i})}{\partial x_i} v_i.$$

Its magnitude, $\left|\partial \widetilde{U}_i(x_i, x_j)/\partial x_i\right|$, indicates one's marginal benefit when adjusting his effort choice to improve his gain-loss utility and thus the strength of the uncertainty-reducing effect. The term boils down to zero in the symmetric case because $p_i(x_i, \boldsymbol{x}_{-i}) = 1/2$: The uncertainty-reducing effect vanishes on the margin, and the competition effect thus fades away. Contestants thus behave as if they are under standard preferences, which leads to the prediction of part (i) of Proposition 3, as in Gill and Stone (2010, Proposition 2) and Dato et al. (2017, Proposition 1).

When contestants are heterogeneous, the CPNE deviates from the Nash equilibrium under standard preferences. Note that both the uncertainty-reducing and competition effects weaken the underdog's effort incentive, and hence he would reduce effort unambiguously. In contrast, the strong contestant's effort choice is subject to competing forces. He is inclined to increase his effort to reduce uncertainty, while he is tempted to slack off because of the weaker opponent's concession. As Proposition 3(ii) indicates, the tension depends on the degree of asymmetry in the competition. The favorite would decrease his effort if and only if the distribution of prize valuations is excessively dispersed, i.e., $v_1/v_2 > 3$: In that case, the competition effect for the favorite suffices to overturn the uncertainty-reducing effect.

Contestants tend to be more sensitive in response to changes in their opponents' actions when the contest is more asymmetric. Fig. 2 depicts contestants' best responses in three scenarios, with and without loss aversion. The best-response correspondence is concave, which implies stronger strategic dependence when x_1/x_2 is large; conversely, it vanishes when efforts are close. It is thus intuitive to conclude that the competition effect is magnified when the contest is more asymmetric. We then obtain Proposition 3(ii).

3.2. Contests with three or more contestants: $N \geq 3$

We now extend the analysis to contests with three or more contenders. We first consider the symmetric case with $v_i = v > 0$ for all $i \in \mathcal{N}$. In contrast to the symmetric two-player contest, the CPNE departs from the Nash equilibrium under standard preferences.

Proposition 4. *Suppose that the contest involves $N \geq 3$ homogeneous contestants with $v_1 = \cdots = v_N =: v > 0$ for all $i \in \mathcal{N}$. When Assumption 2 is satisfied and $k \in [0, \frac{1}{3}]$, a unique symmetric CPNE exists, in which all contestants exert an effort $x^*(k)$, with*

$$x^*(k) = \frac{N-1}{N^2}\left(1 - \frac{N-2}{N}k\right)v.$$

A contestant's equilibrium effort strictly decreases with k, i.e., $dx^*(k)/dk < 0$.

Proposition 4 states that with three or more homogeneous contestants, loss aversion always weakens effort incentives: A contestant's equilibrium effort $x^*(k)$ strictly decreases with k. The contrast with the observation obtained in the symmetric two-player case reveals the nuance caused by additional players. To see this, recall that the impact of loss aversion fades away in the two-player case because each contestant wins with a probability of $1/2$. In a multi-player case, every contestant is technically an underdog despite the symmetry: One wins with a probability of $1/N$ and behaves as if he were competing against an opponent who bids $N-1$ times as much as he does. The uncertainty-reducing effect compels all contestants to decrease their efforts, as the single underdog does in the asymmetric two-player contest. However, the competition effect differs from the bilateral case. Each contestant i responds to $\sum_{j \neq i} x_j$, which amounts to $(N-1)x_i$ for a symmetric effort profile. The competition effect encourages each contestant to step up efforts, since $\sum_{j \neq i} x_j$ decreases due to the uncertainty-reducing effect. However, Proposition 4 shows that the competition effect does not suffice to reverse the uncertainty-reducing effect.

We then proceed to the more complex case of asymmetric players. The equilibrium analysis is complicated by the fact that a player may choose to stay inactive—i.e., exerting zero effort—in which case a corner equilibrium arises. A closed-form solution to the equilibrium with expectations-based loss aversion—i.e., $k > 0$—is in general unavailable. However, we verify that contestants' equilibrium efforts are well behaved, which allows us to conduct comparative statics of k at $k = 0$. The observations suffice to demonstrate the subtle incentive effects imposed by loss aversion in the extended setting.

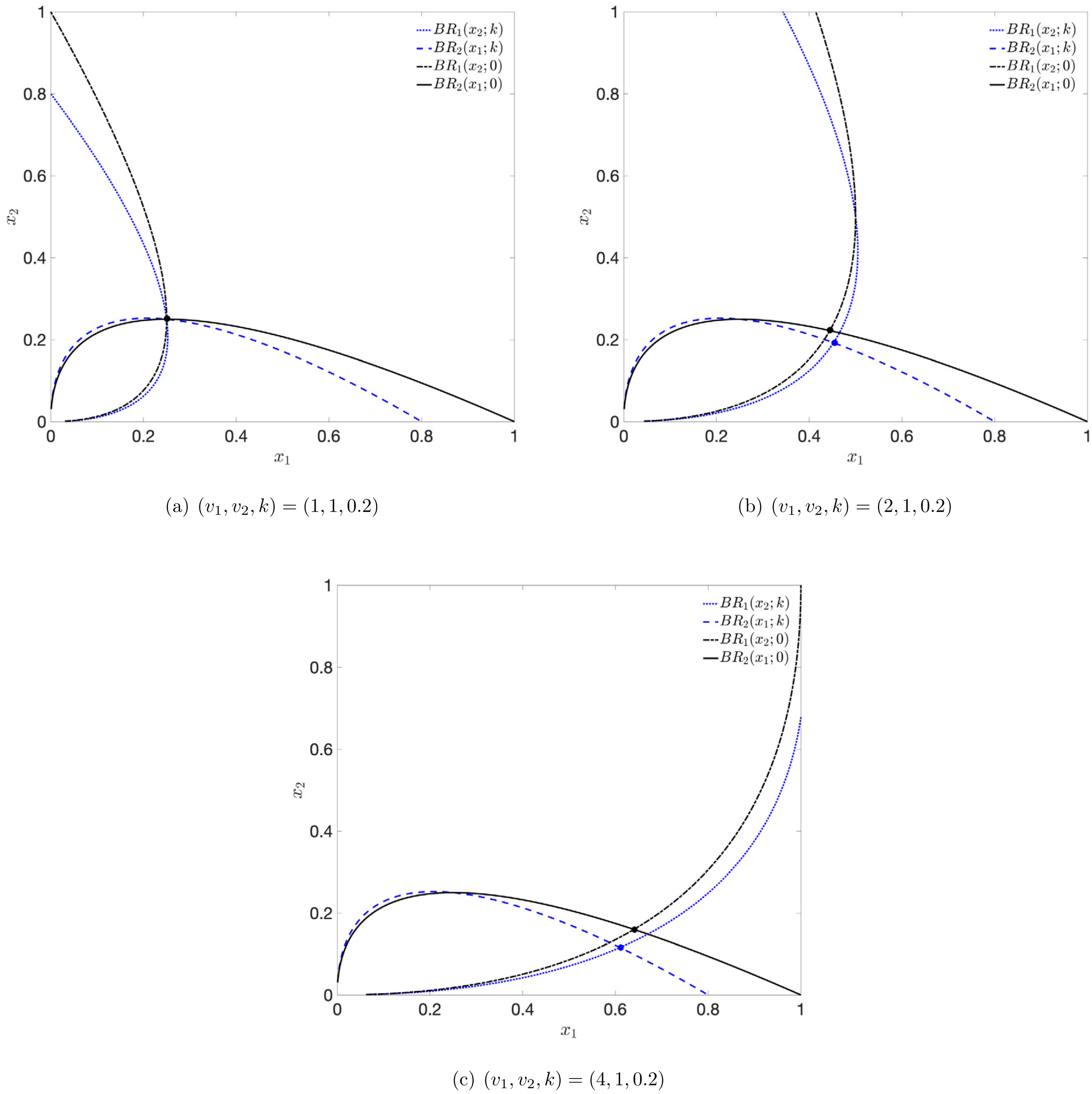
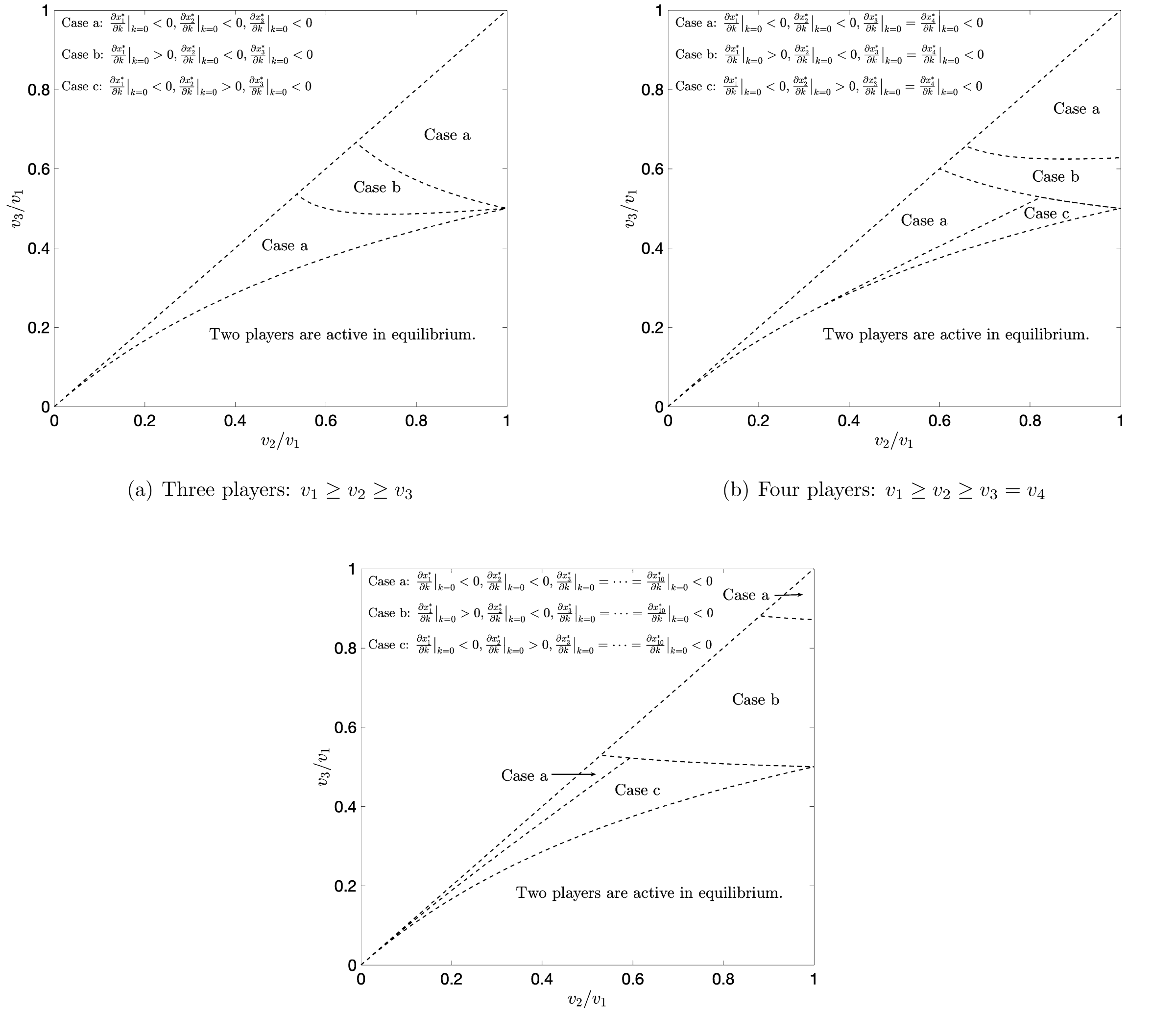


Fig. 2. Equilibrium Effort Profiles: $\left(x_1^*(k), x_2^*(k)\right)$ and $\left(x_1^*(0), x_2^*(0)\right)$.

Denote by $\mathcal{M}(k)$ the set of active players under parameter k. Proposition 2 implies that a weaker contestant must have resigned before a stronger one does, and thus the set of active players is $\mathcal{M}(k) = \left\{1, \ldots, \left|\mathcal{M}(k)\right|\right\}$. For notational convenience, let $m := \left|\mathcal{M}(0)\right|$. In other words, m is the number of active players when contestants have standard preferences (i.e., $k = 0$). The following proposition can be obtained.

Proposition 5. *Suppose that $N \geq 3$ and Assumption 2 is satisfied. Then one of the following three possibilities regarding $\frac{d\boldsymbol{x}^*}{dk}\Big|_{k=0} \equiv \left(\frac{dx_1^*}{dk}\Big|_{k=0}, \ldots, \frac{dx_N^*}{dk}\Big|_{k=0}\right)$ must hold:*

(a) $\frac{dx_i^*}{dk}\Big|_{k=0} \leq 0$ *for all $i \in \mathcal{N}$;*
(b) *There exists a cutoff $\tau_x \in \{1, \ldots, m-1\}$ such that $\frac{dx_i^*}{dk}\Big|_{k=0} > 0$ for $i \in \{1, \ldots, \tau_x\}$ and $\frac{dx_i^*}{dk}\Big|_{k=0} \leq 0$ otherwise;*
(c) *There exists a cutoff $\widehat{\tau}_x \in \{2, \ldots, m-1\}$ such that $\frac{dx_1^*}{dk}\Big|_{k=0} < 0$, $\frac{dx_i^*}{dk}\Big|_{k=0} \geq 0$ for $i \in \left\{2, \ldots, \widehat{\tau}_x\right\}$, and $\frac{dx_i^*}{dk}\Big|_{k=0} \leq 0$ otherwise.*



(a) Three players: $v_1 \geq v_2 \geq v_3$

(b) Four players: $v_1 \geq v_2 \geq v_3 = v_4$

(c) Ten players: $v_1 \geq v_2 \geq v_3 = \cdots = v_{10}$

Fig. 3. Impact of Reference-dependent Preferences on Player Incentives.

The incentive effect of loss aversion depends on the profile of contestants' prize valuations and the number of contestants. Despite the complexity, Proposition 5 states that three patterns are possible. In case (a), all contestants decrease their efforts. Equilibrium efforts bifurcate in case (b), in that strong contestants step up their efforts and weaker contestants do the opposite. Case (c) instead depicts a nonmonotone pattern: A set of middle-ranked contestants—i.e., $\{2, \ldots, \widehat{\tau}_x\}$—increase their bids and the rest are all discouraged, including the top contender (i.e., contestant 1).

We now illustrate these results and elaborate on the underlying logic. The rationale based on the tension between the uncertainty-reducing and competition effects continues to apply to multi-player settings.

Three-contestant scenario: Fig. 3(a) We begin with a three-contestant setting and illustrate it in Fig. 3(a). The horizontal axis represents the ratio v_2/v_1 and the vertical axis v_3/v_1, with both ranging from 0 to 1. The area below the diagonal encompasses all parameterizations relevant to our model, i.e., with $v_1 \geq v_2 \geq v_3$. The bottom-right region of the figure depicts a situation in which contestant 3 stays inactive. We focus on the scenarios in which all three contestants are active.

Case (a) of Proposition 5 takes place in the lower portion of the relevant region, with all contestants decreasing their efforts. This observation is in parallel to the finding of Proposition 3(ii) in the highly asymmetric two-player contest. In this case, v_2 and/or v_3 are small compared with v_1, which allows contestant 1 to dominate the competition. Contestant 1 is tempted to slack off, by the competition effect, when his two (weak) opponents concede; this more than offsets the uncertainty-reducing effect, as in the two-player setting.

Case (b) arises in the middle portion of the relevant region, which represents a scenario of a less lopsided competition, in the sense that the prize valuations of contestants 2 and 3 are closer to v_1. An analogy can be drawn between this observation and that of Proposition 3(ii) for a mildly asymmetric two-player contest, with equilibrium efforts bifurcating between the strong and the weak.[13]

Case (a) is revived in the upper portion of the diagonal, in which cases v_2 and v_3 are closer to v_1, so a more even race is in place. The observation here, however, is driven by a different force than that in the case of large asymmetry, i.e., the lower portion of the relevant region. Analogous to the situation depicted in Proposition 4 (symmetric multi-player contests), contestant 1 should be viewed, technically, as an underdog despite his moderate advantage over each individual opponent. The uncertainty-reducing effect, which outweighs the competition effect, leads all contestants to reduce their efforts.

As can be seen in Fig. 3, case (c) cannot arise in a three-player contest. This observation can formally be established.

Corollary 2. *Suppose that $N = 3$ and Assumption 2 is satisfied. Then either case (a) or case (b) holds; case (c) never occurs.*

Next, we consider a setting with four contestants. We demonstrate that case (c) is likely and interpret the intuition.

Four-contestant scenario: Fig. 3(b) We now add an additional player to the setting. We assume that the fourth contestant is homogeneous to contestant 3, i.e., $v_1 \geq v_2 \geq v_3 = v_4$, which allows us to fit the four-player scenario into the two-dimensional diagram. Similar to Fig. 3(a), the area below the diagonal in Fig. 3(b) is a full collection of the parameterizations relevant to the setting. Contestants 3 and 4 must employ the same strategy in the equilibrium. As a result, the set of parameterizations that cause contestant 3 to remain inactive in the three-contestant scenario are identical to those that lead both contestants 3 and 4 to drop out of the competition in the current one. Again, we focus on the region in which all contestants remain active and exert positive efforts.

Case (c) of Proposition 5 is now possible with the additional player. The observation can again be interpreted by the tension between the uncertainty-reducing and competition effects. The interaction of the two inherently nonmonotone effects ultimately gives rise to the nonmonotone pattern.

A close inspection of Fig. 3(b) reveals that the ascent of case (c) requires a strong contestant 1 and weak contestants 3 and 4: Contestant 1, as the favorite, wins with a probability more than 1/2; contestants 3 and 4 are marginal contenders with only slim winning odds, so the addition of contestant 4 does not challenge contestant 1's dominance.

First, loss aversion compels contestants 3 and 4 to reduce their efforts to lower uncertainty. Note that the gain-loss utility, defined as $\widetilde{U}_i(x_i, \boldsymbol{x}_{-i}) := -kp_i(x_i, \boldsymbol{x}_{-i})[1 - p_i(x_i, \boldsymbol{x}_{-i})]v_i$, is concave in $p_i(x_i, \boldsymbol{x}_{-i})$, which implies a more pronounced uncertainty-reducing effect for a smaller $p_i(x_i, \boldsymbol{x}_{-i})$. The uncertainty-reducing effect strengthens substantially for contestant 3 when contestant 4 joins the competition, since this substantially lowers his winning odds. We can then expect relatively more significant effort reduction from the weakest contestant(s), which more than doubles that from contestant 3 in the three-contestant setting.

The effort reduction from contestants 3 and 4 further triggers the competition effect, as the other contestants will respond. We now expound on the disparate responses from contestants 1 and 2. Recall that both the uncertainty-reducing and competition effects are nonmonotone in nature—depending critically on a contestant's relative standing in the competition—which causes the complexity.

Contestant 2, as an underdog in this case, is compelled to reduce his effort by the uncertainty-reducing effect. However, more significant effort reduction by the weakest contestants, caused by the addition of the fourth contestant, strengthens the competition effect, which may more than offset the uncertainty-reducing effect on contestant 2 and lead him to increase his effort instead. In contrast, contestant 1, as the favorite, increases his effort to reduce uncertainty. The competition effect nevertheless requires that he reduce his effort. The amplified competition effect overturns the uncertainty effect and leads contestant 1 to slack off.

Ten-contestant scenario: Fig. 3(c) This rationale will further unfold in Fig. 3(c). It depicts a situation in which six more contestants identical to contestants 3 and 4 are included in the competition, with $v_1 \geq v_2 \geq v_3 = \cdots = v_{10}$. The area for case (c) increases, compared with Fig. 3(b). The addition of weak contestants further strengthens the uncertainty-reducing effects on the set of marginal contenders, which, in turn, amplifies the competition effect on the two stronger contestants; competition effects are more likely to overturn the respective uncertainty-reducing effects on them.

[13] This case, as well as the scenario of moderate asymmetry in the two-player case, resembles the bifurcation effect observed in Lange and Ratan (2010) and Balzer and Rosato (2021). Both consider auctions in which each bidder's type (private valuation or private signal for the object's value) is independently and identically distributed. They show that with loss aversion, a bidder of a high type tends to bid more aggressively than the risk-neutral benchmark, while one of a low type bids less. However, the similar observations are driven by different mechanisms. In our model, the bifurcation stems from (i) the uncertainty caused by the stochastic winner-selection mechanism and (ii) the direct strategic interaction between heterogeneous contestants and the resultant nonmonotone competition effect. In contrast, in an auction model, ex ante symmetric bidders face opponents of unknown types, which causes the uncertainty and, in turn, the bifurcating responses by bidders of different types.

3.3. Equilibrium outcome

Despite the mixed responses to loss aversion in terms of individual efforts, we can obtain unambiguous predictions regarding its impact on equilibrium outcomes, i.e., the set of active contestants, total effort, and equilibrium winning probability profile.

Active contestants Recall that $\mathcal{M}(k)$ refers to the set of active players under loss aversion k and $\mathcal{M}(k) = \left\{1, \dots, \left|\mathcal{M}(k)\right|\right\}$. The following can be obtained.

Proposition 6. *Suppose that Assumption 2 is satisfied, $k \in [0, \frac{1}{3}]$, and $N \geq 2$. Then $\mathcal{M}(k) \subseteq \mathcal{M}(0)$ and thus $\left|\mathcal{M}(k)\right| \leq \left|\mathcal{M}(0)\right|$.*

Proposition 6 states that whenever a pure-strategy equilibrium exists, i.e., $k \in [0, \frac{1}{3}]$, expectations-based loss aversion always leads to a smaller set of active contestants. That is, weak contestants are more likely to drop out of the competition when they are subject to loss aversion. The intuition is straightforward. It can be verified that $\left|\partial \widetilde{U}_i(x_i, \boldsymbol{x}_{-i})/\partial x_i\right|$ is decreasing when x_i falls below $\sum_{j \neq i} x_j$, which implies that the uncertainty-reducing effect discourages relatively weaker contestants more significantly.

Total effort Proposition 5 shows that three possible patterns can be observed in response to loss aversion for individual equilibrium efforts. However, its impact on total effort—i.e., $\sum_{i=1}^{N} x_i^*$—is clear-cut.

Proposition 7. *Suppose that $N \geq 2$ and Assumption 2 is satisfied. The following statements hold:*

(i) *If $\left|\mathcal{M}(0)\right| = 2$ and $v_1 = v_2 =: v$, then $\sum_{i=1}^{N} x_i^*(k) = \frac{1}{2} v$ for $k \in [0, \frac{1}{3}]$ and hence $\sum_{i=1}^{N} \frac{dx_i^*}{dk}\Big|_{k=0} = 0$;*
(ii) *Otherwise, $\sum_{i=1}^{N} \frac{dx_i^*}{dk}\Big|_{k=0} < 0$.*

Consistent with Proposition 6, Proposition 7 affirms the overall discouraging role played by expectations-based loss aversion. Although loss aversion triggers disparate effort responses from heterogeneous contestants, total effort always decreases except for the knife-edge case in which two homogeneous top contenders remain active in the competition: The game degenerates to the symmetric two-player contest in which the impact of loss aversion disappears.

Equilibrium winning probabilities Denote the equilibrium winning probability profile by $\boldsymbol{p}^* := (p_1^*, \dots, p_N^*)$. Note that $\mathcal{M}(k) = \mathcal{M}(0)$ for small k. The following result can be obtained.

Proposition 8. *Suppose that $N \geq 2$ and Assumption 2 is satisfied. The following statements hold:*

(i) *If $v_1 = \cdots = v_{\mathcal{M}(0)} =: v$, then $p_i^* = \frac{1}{\mathcal{M}(0)}$ for all $i \in \mathcal{M}(0)$ and $k \in [0, \frac{1}{3}]$;*
(ii) *If $v_1 \geq \cdots \geq v_{\mathcal{M}(0)}$, with strict inequality holding for at least one, then there exists a cutoff $\tau_p \in \{1, \dots, N-1\}$ such that $\frac{dp_i^*}{dk}\Big|_{k=0} > 0$ for $i \leq \tau_p$ and $\frac{dp_i^*}{dk}\Big|_{k=0} \leq 0$ for $i > \tau_p$.*

Despite the mixed patterns of changes in effort incentives $(dx_i^*/dk)\Big|_{k=0}$, loss aversion causes winning probabilities to bifurcate between strong and weak contestants. A more elitist distribution pattern results, as winning odds are increasingly concentrated on the top contestants.

4. Discussion and extension

In Section 3, we assume loss-averse contestants à la Kőszegi and Rabin (2006, 2007) and adopt the solution concept of CPNE. Next, we consider two variations. We first consider a contest with risk-averse contestants. We then consider another popularly adopted equilibrium notion, the *preferred personal Nash equilibrium* (PPNE).

4.1. Loss aversion vs. risk aversion

In this part, we analyze the contest game in which we assume risk-averse contestants. We then compare the equilibrium prediction with that with loss-averse contestants. To simplify the analysis, we follow the literature (e.g., Skaperdas and Gan, 1995; Cornes and Hartley, 2003a; Jindapon and Yang, 2017; March and Sahm, 2018) and assume that contestants' preference exhibits constant absolute risk aversion (CARA), with a utility function

$$u(c) = \frac{1-\exp(-\gamma c)}{\gamma}, \text{ with } \gamma > 0. \tag{5}$$

The parameter γ can be interpreted as a measure of the degree of risk aversion. Risk neutrality is a special case, because a linear utility, $u(c) = c$, would result when γ approaches 0.

Effort is costly and depletes a contestant's wealth at a unitary price. Fixing an effort profile $\boldsymbol{x} \equiv (x_1, \dots, x_N)$, contestant $i \in \mathcal{N} \equiv \{1, \dots, N\}$ ends up with a wealth of $v_i - x_i$ if he wins—which occurs with probability $p_i(x_i, \boldsymbol{x}_{-i})$—and $-x_i$ if he loses—which occurs with probability $1 - p_i(x_i, \boldsymbol{x}_{-i})$. His expected utility is given by

$$\mathcal{U}_i(x_i, \boldsymbol{x}_{-i}) := p_i(x_i, \boldsymbol{x}_{-i}) u(v_i - x_i) + \left[1 - p_i(x_i, \boldsymbol{x}_{-i})\right] u(-x_i).$$

Yamazaki (2009) shows that a unique pure-strategy equilibrium exists with heterogeneous risk-averse contestants under nonincreasing absolute risk-aversion preferences.[14] His result can readily be applied to our setting with CARA preferences. Denote by $\boldsymbol{x}^\star(\gamma) := \left(x_1^\star(\gamma), \dots, x_N^\star(\gamma)\right)$ and $\mathcal{M}^\star(\gamma)$ the unique equilibrium effort profile and the set of active players under risk-aversion parameter γ, respectively. The following result in parallel to Proposition 2 can be obtained, which fully characterizes the equilibrium in the contest game.

Proposition 9. *Suppose that contestants are risk averse with an identical utility function in the form of (5) and Assumption 2 is satisfied. Then contestant i's effort in the unique pure-strategy equilibrium $\boldsymbol{x}^\star(\gamma) \equiv \left(x_1^\star(\gamma), \dots, x_N^\star(\gamma)\right)$, with $i \in \mathcal{N}$, is given by*

$$x_i^\star(\gamma) = \begin{cases} \frac{|\mathcal{M}^\star(\gamma)|-1}{\gamma(1-e^{-\gamma v_i})} \times \left[\frac{1}{\left(\sum_{j=1}^{|\mathcal{M}^\star(\gamma)|} \frac{1}{1-e^{-\gamma v_j}}\right)-1} - \frac{e^{-\gamma v_i}}{\sum_{j=1}^{|\mathcal{M}^\star(\gamma)|} \frac{e^{-\gamma v_j}}{1-e^{-\gamma v_j}}} \right] & \text{if } i \in \mathcal{M}^\star(\gamma), \\ 0 & \text{if } i \in \mathcal{N} \setminus \mathcal{M}^\star(\gamma), \end{cases}$$

where $\mathcal{M}^\star(\gamma) = \left\{1, \dots, |\mathcal{M}^\star(\gamma)|\right\}$ and the number of active players—i.e., $|\mathcal{M}^\star(\gamma)|$—is given by

$$|\mathcal{M}^\star(\gamma)| = \max \left\{ n = 2, \dots, N \,\middle|\, e^{-\gamma v_n} < \frac{\sum_{j=1}^{n} \frac{e^{-\gamma v_j}}{1-e^{-\gamma v_j}}}{\left(\sum_{j=1}^{n} \frac{1}{1-e^{-\gamma v_j}}\right) - 1} \right\}.$$

The equilibrium result allows us to conduct the comparative static analysis of risk aversion. We first consider a two-player setting. Denote by $\left(p_1^\star(\gamma), p_2^\star(\gamma)\right)$ the equilibrium winning probability profile under risk aversion. The following result ensues.

Proposition 10. *Suppose that contestants are risk averse with an identical utility function in the form of (5). Further, Assumption 2 is satisfied and $N = 2$. The following statements hold:*

(i) *If $v_1 = v_2 =: v$, then $x_1^\star(\gamma) = x_2^\star(\gamma) = \frac{1}{2\gamma} \times \frac{1-e^{-\gamma v}}{1+e^{-\gamma v}}$, and hence $\lim_{\gamma \searrow 0} \frac{dx_1^\star}{d\gamma} = \lim_{\gamma \searrow 0} \frac{dx_2^\star}{d\gamma} = 0$, $\lim_{\gamma \searrow 0} \frac{d^2 x_1^\star}{d\gamma^2} = \lim_{\gamma \searrow 0} \frac{d^2 x_2^\star}{d\gamma^2} < 0$, and $p_1^\star(\gamma) = p_2^\star(\gamma) = \frac{1}{2}$.*

(ii) *If $v_1 > v_2$, then $\lim_{\gamma \searrow 0} \frac{dx_1^\star}{d\gamma} > 0 > \lim_{\gamma \searrow 0} \frac{dx_2^\star}{d\gamma}$, and hence $\lim_{\gamma \searrow 0} \frac{dp_1^\star}{d\gamma} > 0 > \lim_{\gamma \searrow 0} \frac{dp_2^\star}{d\gamma}$. Moreover, $\lim_{\gamma \searrow 0} \sum_{i=1}^{2} \frac{dx_i^\star}{d\gamma} = 0$ and $\lim_{\gamma \searrow 0} \sum_{i=1}^{2} \frac{d^2 x_i^\star}{d\gamma^2} < 0$.*

The result stands in contrast to the prediction obtained under loss aversion. First, recall, by Proposition 3(i), that in a two-player symmetric contest, the equilibrium coincides with the NE and the impact of loss aversion vanishes. By Proposition 10(i), risk aversion leads symmetric contestants to decrease their efforts. Second, by Proposition 10(ii), heterogeneous contestants' equilibrium efforts bifurcate under risk aversion, while the occurrence of bifurcation under loss aversion requires moderate heterogeneity by Proposition 3(ii).

Recall that the number of active players under risk neutrality is m. We further proceed to a setting of three or more risk-averse contestants and obtain the following.

Proposition 11. *Suppose that contestants are risk averse with an identical utility function in the form of (5). Further, Assumption 2 is satisfied and $N \geq 3$. If $v_1 \geq \cdots \geq v_m$, then one of the following two possibilities regarding $\left(\lim_{\gamma \searrow 0} \frac{dx_1^\star}{d\gamma}, \dots, \lim_{\gamma \searrow 0} \frac{dx_N^\star}{d\gamma}\right)$ must hold:*

(a) *$\lim_{\gamma \searrow 0} \frac{dx_i^\star}{d\gamma} \leq 0$ for all $i \in \mathcal{N}$;*

[14] See also Cornes and Hartley (2012a).

(b) *There exists a cutoff* $\tau_x^\star \in \{1, \ldots, m-1\}$ *such that* $\lim_{\gamma \searrow 0} \frac{dx_i^\star}{d\gamma} > 0$ *for* $i \in \left\{1, \ldots, \tau_x^\star\right\}$ *and* $\lim_{\gamma \searrow 0} \frac{dx_i^\star}{d\gamma} \leq 0$ *otherwise.*

Proposition 11 shows that in equilibrium, contestants' effort responses to risk aversion may exhibit two possible patterns. Both—i.e., (a) all decreasing efforts and (b) bifurcation in contestants' efforts—arise in the case of loss aversion. However, by Proposition 5, a third pattern may arise under loss aversion: The strongest contestant and the set of the weakest decrease their efforts, while middle contestants step up their bids. Such a nonmonotone response pattern, by Proposition 11, does not arise in the setting of risk aversion.

The contrasting predictions demonstrate the different role played by loss aversion than that of risk aversion. To interpret the observations, we first rewrite a risk-averse contestant's expected utility as

$$\begin{aligned}\mathcal{U}_i(x_i, \boldsymbol{x}_{-i}) &:= p_i(x_i, \boldsymbol{x}_{-i}) u(v_i - x_i) + \left[1 - p_i(x_i, \boldsymbol{x}_{-i})\right] u(-x_i) \\ &= p_i(x_i, \boldsymbol{x}_{-i}) \left[u(v_i - x_i) - u(-x_i)\right] + u(-x_i).\end{aligned}$$

The contestant's effort choice, x_i, determines his utility for losing, i.e., $u(-x_i)$, his winning probability $p_i(x_i, \boldsymbol{x}_{-i})$ for given x_{-i}, and the utility differential between winning and losing, i.e., $u(v_i - x_i) - u(-x_i)$.

In the limiting case of risk neutrality (linear utility function), the expected utility can be written as

$$p_i(x_i, \boldsymbol{x}_{-i}) \left[(v_i - x_i) - (-x_i)\right] + (-x_i).$$

The third channel is muted because $(v_i - x_i) - (-x_i)$ boils down to v_i, which is independent of the effort choice x_i and differentiates the decision problem under risk neutrality from that under risk aversion.

We now turn to a loss-averse contestant, in which case the expected payoff is

$$\widehat{U}_i(\boldsymbol{x}) := \left[p_i(x_i, \boldsymbol{x}_{-i}) v_i - x_i\right] - k p_i(x_i, \boldsymbol{x}_{-i})[1 - p_i(x_i, \boldsymbol{x}_{-i})] v_i.$$

In contrast to his risk-neutral counterpart, the effort choice affects not only his winning probability and the material utility for losing, but also the gain-loss utility $-kp_i(x_i, \boldsymbol{x}_{-i})[1 - p_i(x_i, \boldsymbol{x}_{-i})]v_i$.

Decomposition of expected utility functions unveils the fundamental difference between risk aversion and loss aversion in the resultant decision problems. As O'Donoghue and Sprenger (2018) point out, risk aversion is primarily driven by the concern about the distribution over wealth states, while loss aversion is defined by gains and losses relative to references. A risk-averse contestant is concerned about consumption smoothing and thus consciously manipulates his consumption levels across states, which determines the *utility differential* $[u(v_i - x_i) - u(-x_i)]$ between win and loss.

Loss aversion generates an alternative decision problem. A contestant is concerned about the *probabilistic distribution of his consumption* and thus manages the outcome of the contest—i.e., his winning odds $p_i(x_i, \boldsymbol{x}_{-i})$—to reduce the pain inflicted by the uncertainty as measured by $p_i(x_i, \boldsymbol{x}_{-i})[1 - p_i(x_i, \boldsymbol{x}_{-i})]$. It is important to note that the gain-loss utility is nonmonotone in $p_i(x_i, \boldsymbol{x}_{-i})$, which reflects its reference-dependent nature. As a result, contestants of different standing in the competition respond differently in their effort choices because of their disparate expectations. Such nonmonotonicity does not arise in a risk-averse setting, which dismisses the nuanced pattern of case (c).

4.2. Alternative equilibrium concept

Dato et al. (2017) and Dato et al. (2018) adapt the notion of *personal equilibrium* (PE) and develop the concept of *personal Nash equilibrium* (PNE) in game-theoretical contexts, which is formally defined as follows.

Definition 2. (Personal Nash equilibrium) The effort profile $\boldsymbol{x}^{**} \equiv (x_1^{**}, \ldots, x_N^{**})$ constitutes a personal Nash equilibrium (PNE) in pure strategy if for all $i \in \mathcal{N}$,

$$U_i(x_i^{**}, x_i^{**}, \boldsymbol{x}_{-i}^{**}) \geq U_i(x_i, x_i^{**}, \boldsymbol{x}_{-i}^{**}), \text{ for all } x_i \in [0, \infty).$$

The concept of PE requires that a contestant's reference point be fixed (i.e., choice unacclimating), and not adjust to his choice of effort when taking action. A PNE further requires that all contestants be willing to follow their credible effort plan. The notions of PE and PNE are arguably more plausible for contexts in which outcomes are realized shortly after players take actions, in that their expectations do not have enough time to adapt to actual decisions and can be considered exogenous.

In contrast to CPNE, contestants with *fixed* expectations under PNE are attached to the amount of effort they expected to sink, and thus there may exist multiple plans a contestant is willing to implement. Multiple equilibria may often arise. To address the issue of multiple equilibria, Kőszegi and Rabin (2006) argue that agents should be expected to choose their most preferred PE, which gives rise to the concept of *preferred personal equilibrium* (PPE) and *preferred personal Nash equilibrium* (PPNE), PPE's game-theoretic variant (Dato et al., 2017; Dato et al., 2018).

Following Dato et al. (2018), denote by $\Theta_i(\boldsymbol{x}_{-i})$ the set of pure-strategy PEs of contestant i for a given effort profile of his opponents, $\boldsymbol{x}_{-i} \equiv (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$. PPNE is formally defined as follows.

Definition 3. (Preferred personal Nash equilibrium) An effort profile $\boldsymbol{x}^{**} \equiv (x_1^{**}, \ldots, x_N^{**})$ constitutes a preferred personal Nash equilibrium (PPNE) in pure strategy if for all $i \in \mathcal{N}$,

$$U_i(x_i^{**}, x_i^{**}, \boldsymbol{x}_{-i}^{**}) \geq U_i(x_i, x_i, \boldsymbol{x}_{-i}^{**}), \text{ for all } x_i \in \Theta_i(\boldsymbol{x}_{-i}^{**}).$$

For a generic contest game, we first obtain the following.

Proposition 12. *There exists a unique pure-strategy PPNE in the contest game if Assumption 1 is satisfied and $k \in [0, \frac{1}{3}]$.*

In parallel to Proposition 1, Proposition 12 establishes the existence and uniqueness of PPNE for moderate loss aversion.[15] Recall that a contestant attaches a weight η to his gain-loss utility relative to the material utility. We further obtain the following.

Proposition 13. *Suppose that Assumption 1 is satisfied and fix $\lambda > 1$. Then there exists a threshold $\tilde{\eta} \in \left(0, \frac{1}{3(\lambda-1)}\right)$ such that for all $\eta < \tilde{\eta}$, the unique CPNE of the contest game coincides with the unique PPNE.*

Proposition 13 states that PPNE is equivalent to CPNE when η is sufficiently small, which implies that the contestant's concern about his gain-loss utility remains tempered: The prediction obtained under Section 3 would remain intact in this case even if PPNE were adopted as the solution concept.

When η exceeds the threshold, the prediction under PPNE may depart from that under CPNE. In Online Appendix C, we demonstrate in two examples that the main results under Section 3 remain robust.

5. Concluding remarks

This paper explores the equilibrium interactions in contests with loss-averse contestants à la Kőszegi and Rabin (2006, 2007). We first establish the existence and uniqueness of CPNE in the contest. We then investigate the incentive effects of loss aversion, as well as its impact on the equilibrium winning probability profile. We demonstrate that loss aversion yields subtle effects on contestants' behavior: It catalyzes a (direct) uncertainty-reducing effect, which further triggers an (indirect) competition effect. The tension between the two effects could lead contestants to either increase or decrease their efforts. Despite the ambiguous impact of expectations-based loss aversion on incentives, we show that its impact on the set of active players, total effort, and equilibrium winning probability profile is clear-cut. Finally, we consider two extensions and show that (i) loss aversion generates different predictions from those of risk aversion, and (ii) the main results are robust under the alternative equilibrium concept of PPNE.

Our analysis also generates empirically relevant implications. Several of our results may inspire future experimental or empirical studies. For instance, Proposition 3(ii) predicts, in a bilateral setting, how the equilibrium effort profile under loss aversion would deviate from that under risk neutrality with respect to the degree of asymmetry in the contest: Bifurcation arises in a moderately asymmetric contest, while efforts drop for both contestants when the contest is excessively lopsided. Such predictions suggest the possibility of empirically identifying the roles played by loss aversion in contest-like settings. For multi-player cases, effort responses to loss aversion vary in a complex manner among contestants. However, Proposition 8(ii) shows clearly that winning probabilities bifurcate for loss-averse contestants. The experimental literature has traditionally focused on contestants' effort choices, whereas our results may open up an avenue for laboratory studies regarding contestants' winning probabilities.

This paper is an early foray into the implications of expectations-based loss aversion in contests. Large room remains for future studies. First, this research stream calls for richer model settings. Recently, Goette et al. (2019) examine the role of heterogeneity in gain-loss attitude in identifying models of expectations-based reference dependence in an individual-decision setting. It is important to examine the competition between contestants who differ in their levels of loss aversion.[16] Our study assumes a complete-information setting. It is intriguing to speculate on how our results may extend when the contest involves incomplete information. It is well known that a lottery contest is in general intractable when contestants' types are privately known. A formal analysis is analytically challenging but worthwhile for further research. Second, many classical questions on contest design can be reexamined assuming loss-averse contestants. Loss aversion may generate novel implications for contest design, since the optimum must internalize the impact of such preferences on contestants' strategies. In a recent study, for instance, Fu et al. (2021b) examine the optimal contest design when contestants are expectations-based loss averse, and demonstrate that the optimum departs from those obtained under standard preferences (risk neutrality or risk aversion). We leave exploration of such possibilities to future research.

[15] Although PPE is uniquely determined in situations of individual decision-making and can be considered to be a reasonable selection criterion, the existence of PPNE cannot always be guaranteed in general. See Dato et al. (2017) for detailed discussions.

[16] In Online Appendix D, we consider a two-player contest and provide a preliminary analysis that allows for heterogeneous loss aversion. We demonstrate that our main predictions do not lose their bite in the alternative setting, but a more comprehensive study is warranted to explore the nuances of such heterogeneity when the number of contestants exceeds two.

Appendix A. Proofs

Proof of Proposition 1

Proof. Define $y_i := f_i(x_i)$, $\boldsymbol{y} := (y_1, \dots, y_N)$, and $s := \sum_{j=1}^N y_j$. Moreover, denote the inverse function of $f_i(\cdot)$ by $\phi_i(\cdot) := f_i^{-1}(\cdot)$. Contestant i's expected payoff in expression (3) can then be rewritten as

$$\widehat{\mathfrak{U}}_i(y_i, \boldsymbol{y}_{-i}) := \frac{y_i}{\sum_{j=1}^N y_j} v_i - k\frac{y_i}{\sum_{j=1}^N y_j} \times \left(1 - \frac{y_i}{\sum_{j=1}^N y_j}\right) v_i - \phi_i(y_i).$$

It can be verified that $\widehat{\mathfrak{U}}_i(\cdot)$ is strictly concave in $y_i > 0$ under Assumption 1 for $k \leq \frac{1}{3}$. Therefore, if $\left.\frac{\partial \widehat{\mathfrak{U}}_i(y_i, \boldsymbol{y}_{-i})}{\partial y_i}\right|_{y_i=0} \leq 0$, or equivalently, $\phi_i'(0)s \geq (1-k)v_i$, then $y_i = 0$. Otherwise, $y_i > 0$ and solves $\frac{\partial \widehat{\mathfrak{U}}_i(y_i, \boldsymbol{y}_{-i})}{\partial y_i} = 0$. Carrying out the algebra, we have

$$\frac{s - y_i}{s^2} \times \left(1 - k + 2k\frac{y_i}{s}\right) = \frac{1}{v_i} \times \phi_i'(y_i). \tag{A.1}$$

Note that $s = 0$ cannot arise in equilibrium. Otherwise, $x_1 = \cdots = x_N = 0$ and a contestant has strict incentive to deviate by increasing his effort $x_i = 0$ to a sufficiently small positive amount. For all $s > 0$, let us define

$$g_i(s) = \begin{cases} 0 & \text{if } (1-k)v_i \leq \phi_i'(0)s, \\ \text{unique positive solution to } \frac{s-y_i}{s^2}\left(1 - k + 2k\frac{y_i}{s}\right) = \frac{1}{v_i}\phi_i'(y_i) & \text{otherwise.} \end{cases} \tag{A.2}$$

From the above analysis, the effort profile $\boldsymbol{x} \equiv (x_1^*, \dots, x_N^*)$ constitutes a CPNE if and only if $\sum_{i=1}^N g_i(s) = s$, or equivalently, $\chi(s) := \sum_{i=1}^N \frac{g_i(s)}{s} - 1 = 0$. Define

$$\rho_i(s) := \frac{g_i(s)}{s}.$$

Then (A.2) indicates that $\rho_i(s) = 0$ for $s \geq \frac{(1-k)v_i}{\phi_i'(0)}$. For $s < \frac{(1-k)v_i}{\phi_i'(0)}$, it follows from (A.1) that

$$(1 - \rho_i) \times \left(1 - k + 2k\rho_i\right) v_i - s \times \phi_i'\left(\rho_i s\right) = 0, \tag{A.3}$$

which in turn implies that

$$\rho_i'(s) = -\frac{\phi_i'\left(\rho_i s\right) + \rho_i s \times \phi_i''\left(\rho_i s\right)}{\left(1 - 3k + 4k\rho_i\right) v_i + s^2 \times \phi_i''\left(\rho_i s\right)}, \tag{A.4}$$

from the implicit function theorem. Because $\phi_i' > 0$ and $\phi_i'' \geq 0$, the numerator on the right-hand side of the above equation is strictly positive. Next, we show that the denominator is strictly positive.

Clearly, we have

$$\left(1 - 3k + 4k\rho_i\right) v_i + s^2 \times \phi_i''\left(\rho_i s\right) \geq 4k\rho_i v_i + s^2 \times \phi_i''\left(\rho_i s\right) > 0,$$

where the first inequality follows from $k \leq \frac{1}{3}$. To complete the proof, it remains to show that $\chi(s) \equiv \sum_{i=1}^N \rho_i(s) - 1 = 0$ has a unique positive solution for the case of $k \leq \frac{1}{3}$.

First, note that $\rho_i(s)$ is strictly decreasing in s for $s \in \left(0, \frac{(1-k)v_1}{\phi_i'(0)}\right)$, and is constant for $s \geq \frac{(1-k)v_1}{\phi_i'(0)}$. It is straightforward to verify that $\rho_i(s)$ is continuous in s and thus $\chi(s)$ is continuous in s. Second, note that

$$\chi\left(\frac{(1-k)v_1}{\phi_i'(0)}\right) = -1.$$

Moreover, it follows from (A.3) that

$$\lim_{s \searrow 0} \rho_i(s) = 1, \text{and } \lim_{s \searrow 0} \chi(s) = N - 1 > 0.$$

Therefore, there exists a unique positive solution to $\chi(s) = 0$, which concludes the proof. □

Proof of Proposition 2

Proof. Note that $f_i(x_i) = x_i$ implies instantly that $y_i = x_i$, and thus $s := \sum_{i=1}^N y_i = \sum_{i=1}^N x_i$. Fixing $k \in \left[0, \frac{1}{3}\right]$, the function $g_i(s)$ defined in (A.2) can be simplified as

$$g_i(s) = \begin{cases} 0 & \text{if } (1-k)v_i \le s, \\ \frac{\sqrt{(1-3k)^2 s^2 + 8ks^2\left(1-k-\frac{s}{v_i}\right)} - (1-3k)s}{4k} & \text{otherwise.} \end{cases}$$

It follows from the proof of Proposition 1 that in the unique CPNE, we must have that $x_i^* = g_i(s)$, where $s > 0$ is the unique solution to $\sum_{i=1}^N g_i(s) = s$. This concludes the proof. □

Proof of Corollary 1

Proof. It follows from the first-order conditions $\left.\frac{\partial \widehat{U}_1(x_1, x_2^*)}{\partial x_1}\right|_{x_1 = x_1^*} = 0$ and $\left.\frac{\partial \widehat{U}_2(x_2, x_1^*)}{\partial x_2}\right|_{x_2 = x_2^*} = 0$ that

$$\frac{x_2^*}{(x_1^* + x_2^*)^2} v_1 - \frac{x_2^*(x_2^* - x_1^*)}{(x_1^* + x_2^*)^3} k v_1 = 1, \tag{A.5}$$

and

$$\frac{x_1^*}{(x_1^* + x_2^*)^2} v_2 - \frac{x_1^*(x_1^* - x_2^*)}{(x_1^* + x_2^*)^3} k v_2 = 1. \tag{A.6}$$

Let $\theta := x_1^*/x_2^*$. Then (A.5) and (A.6) can be rewritten as

$$\frac{1}{1+\theta} v_1 - \frac{1-\theta}{(1+\theta)^2} k v_1 = x_1^* + x_2^*,$$

and

$$\frac{\theta}{1+\theta} v_2 - \frac{\theta(\theta-1)}{(1+\theta)^2} k v_2 = x_1^* + x_2^*.$$

Combining the above two equations yields

$$\frac{1}{1+\theta} v_1 - \frac{1-\theta}{(1+\theta)^2} k v_1 = \frac{\theta}{1+\theta} v_2 - \frac{\theta(\theta-1)}{(1+\theta)^2} k v_2,$$

which is equivalent to

$$(1-k)\theta^2 - \left(\frac{v_1}{v_2} - 1\right) \times (1+k)\theta - \frac{v_1}{v_2}(1-k) = 0. \tag{A.7}$$

Solving for θ, we have that

$$\theta = \frac{1}{2}\left[\left(\frac{v_1}{v_2} - 1\right)\frac{1+k}{1-k} + \sqrt{\left(\frac{v_1}{v_2} - 1\right)^2 \left(\frac{1+k}{1-k}\right)^2 + \frac{4v_1}{v_2}}\right]. \tag{A.8}$$

Substituting (A.8) and $\theta \equiv x_1^*/x_2^*$ into (A.5) and (A.6), we can solve for $x_1^*(k)$ and $x_2^*(k)$ as the following:

$$x_1^*(k) = \frac{\theta}{(1+\theta)^2} v_1 - \frac{\theta(1-\theta)}{(1+\theta)^3} k v_1, \tag{A.9}$$

and

$$x_2^*(k) = \frac{1}{(1+\theta)^2} v_1 - \frac{1-\theta}{(1+\theta)^3} k v_1. \quad \square \tag{A.10}$$

Proof of Proposition 3

Proof. Part (i) of the proposition is trivial, and it remains to prove part (ii). Define $\ell := v_1/v_2 > 1$. In what follows, we add k to θ to emphasize that θ depends on k. It follows from (A.7) and the implicit function theorem that

$$\frac{d\theta(k)}{dk} = \frac{\left[\theta(k)\right]^2 + (\ell-1)\theta(k) - \ell}{2(1-k)\theta(k) - (\ell-1)(1+k)}.$$

Note that $\theta(0)=\ell$ from (A.8). Therefore, we can obtain that

$$\left.\frac{d\theta(k)}{dk}\right|_{k=0}=\frac{\ell^2+(\ell-1)\ell-\ell}{2\ell-(\ell-1)}=\frac{2\ell(\ell-1)}{1+\ell}>0. \tag{A.11}$$

Differentiating $x_1^*(k)$ in (A.9) with respect to k yields

$$\frac{dx_1^*(k)}{dk}=\frac{1-\theta(k)}{\left[1+\theta(k)\right]^3}\times\frac{d\theta(k)}{dk}\times v_1-\frac{\left[\theta(k)\right]^2-4\theta(k)+1}{\left[1+\theta(k)\right]^4}\times\frac{d\theta(k)}{dk}\times kv_1-\frac{\theta(k)\left[1-\theta(k)\right]}{\left[1+\theta(k)\right]^3}v_1;$$

together with (A.11), we have that

$$\left.\frac{dx_1^*(k)}{dk}\right|_{k=0}=\frac{1-\ell}{(1+\ell)^3}\times\frac{2\ell(\ell-1)}{1+\ell}v_1-\frac{\ell(1-\ell)}{(1+\ell)^3}v_1=\frac{\ell(\ell-1)(3-\ell)}{(1+\ell)^4}v_1,$$

from which we can obtain that

$$\left.\frac{dx_1^*(k)}{dk}\right|_{k=0}\gtreqless 0\Leftrightarrow \ell\lesseqgtr 3.$$

Next, we show that $\frac{dx_2^*(k)}{dk}<0$ for all $\ell>1$. Differentiating $x_2^*(k)$ in (A.10) with respect to k yields

$$\frac{dx_2^*(k)}{dk}=\frac{1}{\theta(k)}\times\frac{dx_1^*(k)}{dk}-\frac{1}{\left[\theta(k)\right]^2}\times\frac{d\theta(k)}{dk}\times x_1^*(k).$$

Note that $x_1^*(0)=\frac{\ell}{(1+\ell)^2}v_1$ from (A.9). Therefore, we have that

$$\left.\frac{dx_2^*(k)}{dk}\right|_{k=0}=\frac{1}{\ell}\times\frac{\ell(\ell-1)(3-\ell)}{(1+\ell)^4}v_1-\frac{1}{\ell^2}\times\frac{2\ell(\ell-1)}{1+\ell}\times\frac{\ell}{(1+\ell)^2}v_1=-\frac{(\ell-1)(3\ell-1)}{(1+\ell)^4}v_1<0. \quad \square$$

Proof of Proposition 4

Proof. The equilibrium characterization follows immediately from Proposition 2, and is omitted for brevity. □

Proof of Proposition 5

Proof. The following result due to Stein (2002) fully characterizes the equilibrium for the benchmark case $k=0$ without reference-dependent preferences.

Lemma 1. ***(Stein, 2002)*** *Suppose that Assumption 2 is satisfied and $k=0$. Then the equilibrium effort profile, $\boldsymbol{x}^*(0)\equiv\left(x_1^*(0),\ldots,x_N^*(0)\right)$, is given by*

$$x_i^*(0)=\begin{cases} s(0)-\frac{[s(0)]^2}{v_i} & \text{if } i\in\{1,\ldots,m\},\\ 0 & \text{if } i\in\mathcal{N}\setminus\{1,\ldots,m\},\end{cases} \tag{A.12}$$

where m is the number of active players and is given by

$$m=\max\left\{n=2,\ldots,N\,\middle|\,(n-1)\frac{1}{v_n}<\sum_{j=1}^{n}\frac{1}{v_j}\right\}, \tag{A.13}$$

and

$$s(0)\equiv\sum_{i=1}^{N}x_i^*(0)=\frac{m-1}{\sum_{i=1}^{m}\frac{1}{v_i}}. \tag{A.14}$$

We can now prove the proposition. The proof for the case in which $m=2$ follows immediately from Proposition 3. Further, the proof for the case in which $m\geq 3$ and $v_1=\cdots=v_m$ follows immediately from Proposition 4. Therefore, it remains to consider the case in which $m\geq 3$ and $v_1\geq\cdots\geq v_m$, with strict inequality holding for at least one. It can be verified that $\left|\mathcal{M}(k)\right|=\left|\mathcal{M}(0)\right|\equiv m$ for sufficiently small $k>0$.

Denote the equilibrium winning probability profile by $\boldsymbol{p}^* := (p_1^*, \ldots, p_N^*)$. Recall $s \equiv \sum_{i=1}^N x_i^*$. In what follows, we add k to $(x_1^*, \ldots, x_N^*)$, $(p_1^*, \ldots, p_N^*)$, and s to emphasize that they depend on k.

Combining (A.12) and (A.14) yields that

$$p_i^*(0) = \frac{x_i^*(0)}{s(0)} = 1 - \frac{s(0)}{v_i}, \forall i \in \{1, \ldots, m\}. \tag{A.15}$$

Further, the first-order condition $\left.\frac{\partial \widehat{U}_i(x_i, \boldsymbol{x}_{-i}^*)}{\partial x_i}\right|_{x_i = x_i^*} = 0$ [see also (A.3)] can be written as

$$\left[s(k) - x_i^*(k)\right] \times \left[s(k) - 2x_i^*(k)\right] k v_i + \left[s(k)\right]^3 - s(k)\left[s(k) - x_i^*(k)\right] v_i = 0, \forall i \in \{1, 2, \ldots, m\},$$

where $s(k) \equiv \sum_{i=1}^N x_i^*(k) = \sum_{i=1}^m x_i^*(k)$. Differentiating the above equation with respect to k yields the following:

$$\begin{aligned}&\left(\frac{ds(k)}{dk} - \frac{dx_i^*(k)}{dk}\right)\left[s(k) - 2x_i^*(k)\right] k v_i + \left[s(k) - x_i^*(k)\right]\left(\frac{ds(k)}{dk} - 2\frac{dx_i^*(k)}{dk}\right) k v_i \\ &+ \left[s(k) - x_i^*(k)\right]\left[s(k) - 2x_i^*(k)\right] v_i + 3\left[s(k)\right]^2 \frac{ds(k)}{dk} \\ &- \frac{ds(k)}{dk}\left[s(k) - x_i^*(k)\right] v_i - s(k)\left(\frac{ds(k)}{dk} - \frac{dx_i^*(k)}{dk}\right) v_i = 0, \forall i \in \{1, 2, \ldots, m\}.\end{aligned}$$

Evaluating the above equation at $k = 0$, we can obtain

$$\begin{aligned}&\left[s(0) - x_i^*(0)\right] \times \left[s(0) - 2x_i^*(0)\right] + \frac{3}{v_i}[s(0)]^2 \left.\frac{ds(k)}{dk}\right|_{k=0} - \left.\frac{ds(k)}{dk}\right|_{k=0} \times \left[s(0) - x_i^*(0)\right] \\ &- s(0)\left(\left.\frac{ds(k)}{dk}\right|_{k=0} - \left.\frac{dx_i^*(k)}{dk}\right|_{k=0}\right) = 0, \forall i \in \{1, 2, \ldots, m\}.\end{aligned} \tag{A.16}$$

Summing up all the conditions in (A.16) yields

$$(m-3)[s(0)]^2 + 2\sum_{i=1}^m \left[x_i^*(0)\right]^2 + 3[s(0)]^2 \left.\frac{ds(k)}{dk}\right|_{k=0} \sum_{i=1}^m \frac{1}{v_i} - 2(m-1)s(0)\left.\frac{ds(k)}{dk}\right|_{k=0} = 0,$$

from which we can obtain

$$\left.\frac{ds(k)}{dk}\right|_{k=0} = -\frac{(m-3)[s(0)]^2 + 2\sum_{i=1}^m \left[x_i^*(0)\right]^2}{3[s(0)]^2 \sum_{i=1}^m \frac{1}{v_i} - 2(m-1)s(0)}. \tag{A.17}$$

The above equation can be rewritten as

$$\begin{aligned}\left.\frac{ds(k)}{dk}\right|_{k=0} &= -\frac{(m-3)[s(0)]^2 + 2\sum_{i=1}^m \left\{s(0) - \frac{[s(0)]^2}{v_i}\right\}^2}{3[s(0)]^2 \sum_{i=1}^m \frac{1}{v_i} - 2(m-1)s(0)} \\ &= -\frac{(m-3)[s(0)]^2 + 2\sum_{i=1}^m \left\{s(0) - \frac{[s(0)]^2}{v_i}\right\}^2}{3(m-1)s(0) - 2(m-1)s(0)} = s(0) - \frac{2[s(0)]^3 \times \sum_{i=1}^m \frac{1}{v_i^2}}{m-1},\end{aligned} \tag{A.18}$$

where the first equality follows from (A.12) and (A.17); the second and third equalities follow from (A.14).

Combining $p_i^*(k) = x_i^*(k)/s(k)$ and the first-order condition $\left.\frac{\partial \widehat{U}_i(x_i, \boldsymbol{x}_{-i}^*)}{\partial x_i}\right|_{x_i = x_i^*} = 0$ [see also (A.3)], we have that

$$2k\left[p_i^*(k)\right]^2 + (1 - 3k)p_i^*(k) - 1 + k + \frac{s(k)}{v_i} = 0, \forall i \in \{1, \ldots, m\}.$$

Differentiating the above equation with respect to k and rearranging yield

$$\frac{dp_i^*(k)}{dk} = \frac{-2\left[p_i^*(k)\right]^2 + 3p_i^*(k) - 1 - \frac{1}{v_i}\frac{ds(k)}{dk}}{4kp_i^*(k) + 1 - 3k}, \forall i \in \{1, \ldots, m\},$$

which in turn implies that

$$\begin{aligned}\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} &= -2\left[p_i^*(0)\right]^2+3p_i^*(0)-1-\frac{1}{v_i}\times\left.\frac{ds(k)}{dk}\right|_{k=0} &\text{(A.19)}\\ &= -2\left[p_i^*(0)\right]^2+3p_i^*(0)-1-\frac{1}{v_i}\times\left\{s(0)-\frac{2\left[s(0)\right]^3\times\sum_{i=1}^m\frac{1}{v_i^2}}{m-1}\right\}\\ &= -2\left[p_i^*(0)\right]^2+3p_i^*(0)-1-\left[1-p_i^*(0)\right]\times\left\{1-\frac{2\sum_{i=1}^m\frac{1}{v_i^2}}{m-1}\times\left[\frac{m-1}{\sum_{i=1}^m\frac{1}{v_i}}\right]^2\right\}\\ &= -2\left[1-p_i^*(0)\right]\times\left\{1-\frac{(m-1)\sum_{i=1}^m\frac{1}{v_i^2}}{\left[\sum_{i=1}^m\frac{1}{v_i}\right]^2}-p_i^*(0)\right\},\forall i\in\{1,\dots,m\}. &\text{(A.20)}\end{aligned}$$

The second equality follows from (A.18) and the third equality follows from (A.14) and (A.15).

Let

$$\tilde{p}:=1-\frac{(m-1)\sum_{i=1}^m\frac{1}{v_i^2}}{\left[\sum_{i=1}^m\frac{1}{v_i}\right]^2}.$$

It is straightforward to verify that $\tilde{p}<\frac{1}{2}$ for $m\geq 3$. Moreover, we must have $\tilde{p}>0$. Otherwise, $\left.\frac{dp_i^*(k)}{dk}\right|_{k=0}>0$ for all $i\in\mathcal{M}(0)$ from (A.20), and thus $0=\sum_{i=1}^N\left.\frac{dp_i^*(k)}{dk}\right|_{k=0}=\sum_{i=1}^m\left.\frac{dp_i^*(k)}{dk}\right|_{k=0}>0$, a contradiction. A closer look at (A.20) yields

$$\left.\frac{dp_i^*(k)}{dk}\right|_{k=0}>0\Leftrightarrow p_i^*(0)>\tilde{p}\equiv 1-\frac{(m-1)\sum_{i=1}^m\frac{1}{v_i^2}}{\left[\sum_{i=1}^m\frac{1}{v_i}\right]^2}. \tag{A.21}$$

Next, it follows from $x_i^*(k)=p_i^*(k)\times s(k)$ that

$$\begin{aligned}\left.\frac{dx_i^*(k)}{dk}\right|_{k=0} &= \left.\frac{dp_i^*(k)}{dk}\right|_{k=0}\times s(0)+\left.\frac{ds(k)}{dk}\right|_{k=0}\times p_i^*(0)\\ &= \left\{-2\left[p_i^*(0)\right]^2+3p_i^*(0)-1-\frac{1}{v_i}\times\left.\frac{ds(k)}{dk}\right|_{k=0}\right\}\times s(0)+\left.\frac{ds(k)}{dk}\right|_{k=0}\times p_i^*(0)\\ &= -s(0)\left[1-p_i^*(0)\right]\left[1-2p_i^*(0)\right]+\left.\frac{ds(k)}{dk}\right|_{k=0}\times\left[p_i^*(0)-\frac{s(0)}{v_i}\right]\\ &= -s(0)\times\left[1-2p_i^*(0)\right]\times\left[\frac{\left.\frac{ds(k)}{dk}\right|_{k=0}+s(0)}{s(0)}-p_i^*(0)\right]\\ &= -s(0)\times\left[1-2p_i^*(0)\right]\times\left[2\tilde{p}-p_i^*(0)\right] &\text{(A.22)}\\ &= -s(0)\times\left[\frac{2}{v_i}\times\frac{m-1}{\sum_{i=1}^m\frac{1}{v_i}}-1\right]\times\left[\frac{s(0)}{v_i}-\left(\frac{2\left[s(0)\right]^2\times\sum_{i=1}^m\frac{1}{v_i^2}}{m-1}-1\right)\right], &\text{(A.23)}\end{aligned}$$

where the second equality follows from (A.19); the fourth equality from (A.15); the fifth equality from (A.14), (A.18), and the definition of $\tilde{p}$; and the last equality from (A.14), (A.15), and (A.18). Define

$$v^{\ddagger}:=\frac{2(m-1)}{\sum_{i=1}^m\frac{1}{v_i}}>0,$$

and

$$v^{\ddagger\ddagger} := \frac{s(0)}{\frac{2[s(0)]^2 \times \sum_{i=1}^{m} \frac{1}{v_i^2}}{m-1} - 1} = \frac{(m-1)\sum_{i=1}^{m} \frac{1}{v_i}}{2(m-1)\sum_{i=1}^{m} \frac{1}{(v_i)^2} - \left(\sum_{i=1}^{m} \frac{1}{v_i}\right)^2} > 0.$$

Let $\underline{v} := \min\{v^{\ddagger}, v^{\ddagger\ddagger}\}$ and $\overline{v} := \max\{v^{\ddagger}, v^{\ddagger\ddagger}\}$. Equations (A.22) and (A.23) imply that

$$\left.\frac{dx_i^*(k)}{dk}\right|_{k=0} > 0 \Leftrightarrow \min\left\{\frac{1}{2}, 2\tilde{p}\right\} < p_i^*(0) < \max\left\{\frac{1}{2}, 2\tilde{p}\right\}$$
$$\Leftrightarrow \underline{v} < v_i < \overline{v}.$$

If $\underline{v} = \overline{v}$, then $\left.\frac{dx_i^*(k)}{dk}\right|_{k=0} \leq 0$ for all $i \in \mathcal{N}$. Suppose that $\underline{v} < \overline{v}$. The profile of prize valuations $\boldsymbol{v} \equiv (v_1, \ldots, v_m, \ldots, v_N)$ has to belong to one of the following five cases:

Case I: $\boldsymbol{v_1 \leq \underline{v}}$ or $\boldsymbol{v_m \geq \overline{v}}$. Then $\left.\frac{dx_i^*(k)}{dk}\right|_{k=0} \leq 0$ for all $i \in \mathcal{M}(0)$, which corresponds to the pattern described in Proposition 5(a).

Case II: $\boldsymbol{\underline{v} \leq v_m < v_1 \leq \overline{v}}$. Then $\left.\frac{dx_i^*(k)}{dk}\right|_{k=0} \geq 0$ for all $i \in \mathcal{M}(0)$, with strict inequality holding for at least one. This case is impossible because $\left.\frac{ds(k)}{dk}\right|_{k=0} \equiv \sum_{i=1}^{m} \left.\frac{dx_i^*(k)}{dk}\right|_{k=0} < 0$ by Proposition 7.

Case III: $\boldsymbol{v_m \leq \underline{v} < v_1 \leq \overline{v}}$. Then there exists a cutoff of prize valuation above which $\left.\frac{dx_i^*(k)}{dk}\right|_{k=0} > 0$ and below which $\left.\frac{dx_i^*(k)}{dk}\right|_{k=0} \leq 0$. This corresponds to the pattern described in Proposition 5(b).

Case IV: $\boldsymbol{\underline{v} \leq v_m \leq \overline{v} < v_1}$. This implies that $p_i^*(0) > \min\left\{\frac{1}{2}, 2\tilde{p}\right\} > \tilde{p}$, where the last strict inequality follows from $\tilde{p} \in (0, \frac{1}{2})$. Together with (A.21), we have that $\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} > 0$ for $i \in \mathcal{M}(0)$. This in turn implies that $\sum_{i=1}^{N} \left.\frac{dp_i^*(k)}{dk}\right|_{k=0} = \sum_{i=1}^{m} \left.\frac{dp_i^*(k)}{dk}\right|_{k=0} > 0$, which is a contradiction. Therefore, this case is also impossible.

Case V: $\boldsymbol{v_m \leq \underline{v} \leq \overline{v} < v_1}$. If there exists no contestant whose prize valuation lies between $\underline{v}$ and $\overline{v}$, then $\left.\frac{dx_i^*(k)}{dk}\right|_{k=0} \leq 0$ for all $i \in \mathcal{M}(0)$ and again we have the pattern described in Proposition 5(a). Suppose instead there exists some contestant $t \in \{2, \ldots, m-1\}$ such that $v_t \in (\underline{v}, \overline{v})$ and $v_1 > v_t > v_m$. Next, we show that the pattern described in Proposition 5(c) must arise. It suffices to rule out the situation in which $\left.\frac{dx_1^*(k)}{dk}\right|_{k=0} < 0$ and $\left.\frac{dx_2^*(k)}{dk}\right|_{k=0} \leq 0$. We consider the following two sub-cases:

Sub-case (i): $\boldsymbol{\tilde{p} \geq \frac{1}{4}}$. The postulated $v_m \leq \underline{v} < v_t < \overline{v} < v_1$ implies that $p_1^*(0) \geq p_t^*(0) \geq \min\left\{\frac{1}{2}, 2\tilde{p}\right\} = \frac{1}{2}$. Together with the fact that $p_m^*(0) > 0$, we have that $\sum_{i=1}^{N} p_i^*(0) \geq p_1^*(0) + p_t^*(0) + p_m^*(0) > 1$, which is a contradiction.

Sub-case (ii): $\boldsymbol{\tilde{p} < \frac{1}{4}}$. Because $\left.\frac{dx_1^*(k)}{dk}\right|_{k=0} < 0$ and $\left.\frac{dx_2^*(k)}{dk}\right|_{k=0} \leq 0$, we must have $p_1^*(0) \geq p_2^*(0) \geq \max\left\{\frac{1}{2}, 2\tilde{p}\right\} = \frac{1}{2}$. Together with the fact that $p_m^*(0) > 0$, we can obtain that $\sum_{i=1}^{N} p_i^*(0) \geq p_1^*(0) + p_2^*(0) + p_m^*(0) > 1$, which is again a contradiction.

To summarize, there are three possible patterns on $\left(\left.\frac{dx_1^*(k)}{dk}\right|_{k=0}, \ldots, \left.\frac{dx_N^*(k)}{dk}\right|_{k=0}\right)$, as Proposition 5 predicts. This concludes the proof. □

Proof of Corollary 2

Proof. Suppose to the contrary that case (c) occurs for some combination of (v_1, v_2, v_3). It follows immediately that $v_1 > v_2$. Moreover, all three players must stay active under $k = 0$. Otherwise, $\left.\frac{dx_1^*(k)}{dk}\right|_{k=0} < 0$ and $\left.\frac{dx_2^*(k)}{dk}\right|_{k=0} \geq 0$ cannot hold by Proposition 3.

Without loss of generality, we normalize $v_1 = 1$. By Lemma 1, all three contestants remaining active in equilibrium under $k = 0$ requires that

$$\frac{2}{v_i} < \sum_{j=1}^{3} \frac{1}{v_j}, \forall i \in \{1, 2, 3\} \Rightarrow \frac{1}{v_3} < \frac{1}{v_2} + 1. \tag{A.24}$$

Next, it follows from (A.22) that

$$\left.\frac{dx_i^*(k)}{dk}\right|_{k=0} = -s(0) \times \left[1-2p_i^*(0)\right] \times \left[2\tilde{p} - p_i^*(0)\right],$$

where $\tilde{p} = 1 - \frac{2\sum_{j=1}^{3}\frac{1}{v_j^2}}{\left[\sum_{j=1}^{3}\frac{1}{v_j}\right]^2}$, $s(0) = \frac{2}{\sum_{j=1}^{3}\frac{1}{v_j}}$, and $p_i^*(0) = 1 - \frac{s(0)}{v_i}$. Therefore, for case (c) to occur, we must have $2\tilde{p} \le p_2^* \le \frac{1}{2}$, which in turn implies that

$$3\left(\frac{1}{v_3}\right)^2 - \left(\frac{4}{v_2}+2\right) \times \frac{1}{v_3} + \left(\frac{1}{v_2}\right)^2 - \frac{4}{v_2} + 3 \ge 0.$$

However, the above inequality cannot hold. To see this, note that

$$\begin{aligned}
&3\left(\frac{1}{v_3}\right)^2 - \left(\frac{4}{v_2}+2\right) \times \frac{1}{v_3} + \left(\frac{1}{v_2}\right)^2 - \frac{4}{v_2} + 3 \\
<&\left(\frac{1}{v_2}+1\right)\left(\frac{3}{v_2}+3-\frac{4}{v_2}-2\right) + \left(\frac{1}{v_2}\right)^2 - \frac{4}{v_2} + 3 \\
<&4\left(1-\frac{1}{v_2}\right) < 0,
\end{aligned}$$

where the first inequality follows from $\frac{1}{3}(\frac{2}{v_2}+1) < \frac{1}{v_2} \le \frac{1}{v_3}$ and (A.24), and the second inequality from $v_2 < v_1 = 1$. This concludes the proof. □

Proof of Proposition 6

Proof. Suppose to the contrary that $\left|\mathcal{M}(k)\right| \ge \left|\mathcal{M}(0)\right| + 1 \equiv m+1$. Then we have that

$$\begin{aligned}
\sum_{i=1}^{m+1} p_i^*(k) &= \sum_{i=1}^{m+1}\left[\frac{3}{4} - \frac{1}{4k} + \frac{1}{4}\sqrt{\left(1+\frac{1}{k}\right)^2 - 8\frac{s(k)}{kv_i}}\right] \\
&> \sum_{i=1}^{m+1}\left[\frac{3}{4} - \frac{1}{4k} + \frac{1}{4}\sqrt{\left(1+\frac{1}{k}\right)^2 - 8\frac{(1-k)v_{m+1}}{kv_i}}\right] \\
&= \sum_{i=1}^{m+1}\left[\frac{3}{4} - \frac{1}{4k} + \frac{1}{4}\sqrt{\left(1+\frac{1}{k}-4\frac{v_{m+1}}{v_i}\right)^2 + \frac{16v_{m+1}(v_i - v_{m+1})}{v_i^2}}\right] \\
&> \sum_{i=1}^{m+1}\left[\frac{3}{4} - \frac{1}{4k} + \frac{1}{4}\left(1+\frac{1}{k}-4\frac{v_{m+1}}{v_i}\right)\right] \\
&= \sum_{i=1}^{m+1}\left(1-\frac{v_{m+1}}{v_i}\right) = \sum_{i=1}^{m}\left(1-\frac{v_{m+1}}{v_i}\right) \ge \sum_{i=1}^{m}\left(1-\frac{s(0)}{v_i}\right) = 1,
\end{aligned}$$

where the first equality follows from (4); the first inequality from contestant $m+1$'s participation constraint $s(k) < (1-k)v_{m+1}$ in (4); the second inequality from $v_1 \ge \cdots \ge v_{m+1}$; the third inequality from (4) and the fact that contestant $m+1$ is inactive under $k=0$; and the last equality from the rearrangement of (A.14). Clearly, the above inequality contradicts

$$\sum_{i=1}^{m+1} p_i^*(k) \le \sum_{i=1}^{\left|\mathcal{M}(k)\right|} p_i^*(k) \le \sum_{i=1}^{N} p_i^*(k) = 1.$$

This completes the proof. □

Proof of Proposition 7

Proof. Part (i) of the proposition is straightforward. Clearly, we must have $|\mathcal{M}(k)| \geq 2$. Moreover, it follows from Proposition 6 that $m \equiv |\mathcal{M}(0)| = 2$ indicates that $|\mathcal{M}(k)| \leq 2$. Therefore, we must have $|\mathcal{M}(k)| = 2$ and $x_1^*(k) + x_2^*(k) = \frac{1}{2}v$ from Proposition 3(i).

Next, we prove part (ii) of the proposition. The proof for the case in which $m = 2$ and $v_1 > v_2$ is straightforward; and it remains to prove the result for the case in which $m \geq 3$. By Equation (A.17), we have that

$$\sum_{i=1}^{N} \left.\frac{dx_i^*(k)}{dk}\right|_{k=0} = \left.\frac{ds(k)}{dk}\right|_{k=0} = -\frac{(m-3)\,[s(0)]^2 + 2\sum_{i=1}^{m}\left[x_i^*(0)\right]^2}{3\,[s(0)]^2\sum_{i=1}^{m}\frac{1}{v_i} - 2(m-1)s(0)}.$$

It is evident that the numerator is strictly positive for $m \geq 3$. Moreover, we have that

$$3\,[s(0)]^2\sum_{i=1}^{m}\frac{1}{v_i} - 2(m-1)s(0) = 3\left[\frac{m-1}{\sum_{i=1}^{m}\frac{1}{v_i}}\right]^2\sum_{i=1}^{m}\frac{1}{v_i} - 2(m-1)\frac{m-1}{\sum_{i=1}^{m}\frac{1}{v_i}} = \frac{(m-1)^2}{\sum_{i=1}^{m}\frac{1}{v_i}} > 0,$$

where the first equality follows from (A.14). This in turn implies that $\sum_{i=1}^{N}\left.\frac{dx_i^*(k)}{dk}\right|_{k=0} < 0$. □

Proof of Proposition 8

Proof. Part (i) of the proposition is obvious, and it remains to prove part (ii). Recall that $|\mathcal{M}(k)| = |\mathcal{M}(0)| \equiv m$ for sufficiently small $k > 0$. Plugging (A.15) into (A.19) yields

$$\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} = -2\,[s(0)]^2 \times \frac{1}{v_i^2} + \left[s(0) - \left.\frac{ds}{dk}\right|_{k=0}\right] \times \frac{1}{v_i}, \forall i \in \{1, \ldots, m\}. \tag{A.25}$$

Combining (A.14), (A.18), and (A.25), it can be verified that $\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} > 0$ is equivalent to

$$v_i > \frac{\sum_{i=1}^{m}\frac{1}{v_i}}{\sum_{i=1}^{m}\frac{1}{v_i^2}}.$$

Moreover, simple algebra would verify that

$$v_1 > \frac{\sum_{i=1}^{m}\frac{1}{v_i}}{\sum_{i=1}^{m}\frac{1}{v_i^2}}, \text{ and } v_m < \frac{\sum_{i=1}^{m}\frac{1}{v_i}}{\sum_{i=1}^{m}\frac{1}{v_i^2}}.$$

Therefore, there exists a cutoff τ_p such that $\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} > 0$ for $i \leq \tau_p$ and $\left.\frac{dp_i^*(k)}{dk}\right|_{k=0} \leq 0$ otherwise. □

Proof of Proposition 9

Proof. The analysis closely follows that of Stein (2002) and is omitted for brevity. □

Proof of Proposition 10

Proof. Suppose that $N = 2$. By Proposition 9, the equilibrium effort pair $\left(x_1^\star(\gamma), x_2^\star(\gamma)\right)$ can be derived as follows:

$$x_1^\star(\gamma) = \frac{1}{\gamma} \times \frac{\left(1-e^{-\gamma v_2}\right)\left(1-e^{-\gamma v_1}\right)}{1-e^{-\gamma(v_1+v_2)}} \times \frac{\left(1-e^{-\gamma v_1}\right)e^{-\gamma v_2}}{\left(1-e^{-\gamma v_1}\right)e^{-\gamma v_2} + \left(1-e^{-\gamma v_2}\right)e^{-\gamma v_1}},$$

and

$$x_2^\star(\gamma) = \frac{1}{\gamma} \times \frac{\left(1-e^{-\gamma v_2}\right)\left(1-e^{-\gamma v_1}\right)}{1-e^{-\gamma(v_1+v_2)}} \times \frac{\left(1-e^{-\gamma v_2}\right)e^{-\gamma v_1}}{\left(1-e^{-\gamma v_1}\right)e^{-\gamma v_2} + \left(1-e^{-\gamma v_2}\right)e^{-\gamma v_1}}.$$

Part (i) of the proposition is straightforward and it remains to prove part (ii). Suppose that $v_1 > v_2$. From the above equilibrium characterization, we can obtain

$$s^\star(\gamma) := \sum_{i=1}^{2} x_i^\star(\gamma) = \frac{1}{\gamma} \times \frac{\left(1-e^{-\gamma v_2}\right)\left(1-e^{-\gamma v_1}\right)}{1-e^{-\gamma v_2}e^{-\gamma v_1}},$$

and

$$p_1^\star(\gamma) = 1 - p_2^\star(\gamma) = \frac{\left(1 - e^{-\gamma v_1}\right) e^{-\gamma v_2}}{\left(1 - e^{-\gamma v_1}\right) e^{-\gamma v_2} + \left(1 - e^{-\gamma v_2}\right) e^{-\gamma v_1}}.$$

Carrying out the algebra, we have that

$$\lim_{\gamma \searrow 0} s^\star(\gamma) = \frac{v_1 v_2}{v_1 + v_2}, \ \lim_{\gamma \searrow 0} \frac{ds^\star(\gamma)}{d\gamma} = 0, \text{and} \ \lim_{\gamma \searrow 0} \frac{d^2 s^\star(\gamma)}{d\gamma^2} = -\frac{(v_1 v_2)^2}{6(v_1 + v_2)} < 0.$$

Further,

$$\lim_{\gamma \searrow 0} p_1^\star(\gamma) = \frac{v_1}{v_1 + v_2}, \text{and} \ \lim_{\gamma \searrow 0} \frac{dp_1^\star(\gamma)}{d\gamma} = -\lim_{\gamma \searrow 0} \frac{dp_2^\star(\gamma)}{d\gamma} = \frac{v_1 v_2 (v_1 - v_2)}{2(v_1 + v_2)^2} > 0.$$

Note that $x_1^\star(\gamma) = s^\star(\gamma) \times p_1^\star(\gamma)$. Therefore, we have that

$$\lim_{\gamma \searrow 0} \frac{dx_1^\star(\gamma)}{d\gamma} = \lim_{\gamma \searrow 0} \frac{ds^\star(\gamma)}{d\gamma} \times \lim_{\gamma \searrow 0} p_1^\star(\gamma) + \frac{dp_1^\star(\gamma)}{d\gamma} \times \lim_{\gamma \searrow 0} s^\star(\gamma) = \frac{(v_1 v_2)^2 (v_1 - v_2)}{2(v_1 + v_2)^3} > 0.$$

Similarly, it follows from $x_2^\star(\gamma) = s^\star(\gamma) \times p_2^\star(\gamma)$ that

$$\lim_{\gamma \searrow 0} \frac{dx_2^\star(\gamma)}{d\gamma} = \lim_{\gamma \searrow 0} \frac{ds^\star(\gamma)}{d\gamma} \times \lim_{\gamma \searrow 0} p_2^\star(\gamma) + \frac{dp_2^\star(\gamma)}{d\gamma} \times \lim_{\gamma \searrow 0} s^\star(\gamma) = -\frac{(v_1 v_2)^2 (v_1 - v_2)}{2(v_1 + v_2)^3} < 0.$$

This concludes the proof. □

Proof of Proposition 11

Proof. It can be verified that $\lim_{\gamma \searrow 0} \left| \mathcal{M}^\star(\gamma) \right| = m$, where m is the number of active contestants under risk neutrality and is given by (A.13). Further, we can show that players 1 to m exert positive effort and players $m+1$ to N remain inactive for sufficiently small $\gamma > 0$. Therefore, $\lim_{\gamma \searrow 0} \frac{dx_i^\star}{d\gamma} = 0$ for $i \in \{m+1, \dots, N\}$ and it suffices to consider players 1 to m.

From the equilibrium characterization established in Proposition 9, we can obtain that

$$\lim_{\gamma \searrow 0} \frac{dx_i^\star(\gamma)}{d\gamma} = \frac{m-1}{\left[\sum_{j=1}^m \frac{1}{v_j}\right]^2} \times \left[\frac{1}{2} - \frac{1}{v_i} \times \frac{m-1}{\sum_{j=1}^m \frac{1}{v_j}} \right], i \in \{1, \dots, m\}.$$

Evidently, $\lim_{\gamma \searrow 0} \frac{dx_i^\star(\gamma)}{d\gamma}$ decreases with i. Moreover, it is straightforward to verify that (i) $\lim_{\gamma \searrow 0} \frac{dx_m^\star(\gamma)}{d\gamma} < 0$ for $m \geq 3$ and (ii) the sign of $\lim_{\gamma \searrow 0} \frac{dx_1^\star(\gamma)}{d\gamma}$ depends on the distribution of prize valuations $(v_1, \dots, v_m)$. These altogether indicate that $\left(\lim_{\gamma \searrow 0} \frac{dx_1^\star}{d\gamma}, \dots, \lim_{\gamma \searrow 0} \frac{dx_N^\star}{d\gamma}\right)$ has two patterns, as Proposition 11 predicts, which concludes the proof. □

Proof of Proposition 12

Proof. Recall $\boldsymbol{y} \equiv (y_1, \dots, y_N)$ in the proof of Proposition 1. Contestant i's expected utility in expression (2) can be rewritten as

$$\begin{aligned} \mathfrak{U}_i(y_i, \hat{y}_i, \boldsymbol{y}_{-i}) := & \frac{y_i}{\sum_{j=1}^N y_j} v_i \times \left[(1+\eta) + \eta(\lambda - 1) \frac{\hat{y}_i}{\sum_{j \neq i} y_j + \hat{y}_i} \right] \\ & - \phi_i(y_i) + \eta\mu\left(\phi_i(\hat{y}_i) - \phi_i(y_i)\right) - \eta\lambda \frac{\hat{y}_i}{\sum_{j \neq i} y_j + \hat{y}_i} v_i, \end{aligned} \tag{A.26}$$

where $y_i := f_i(x_i)$ and $\hat{y}_i := f_i(\hat{x}_i)$. To prove the existence and uniqueness of PPNE of the original contest game, it suffices to show that there exists a unique PPNE of the modified contest game in which contestant $i \in \mathcal{N}$ chooses $y_i \geq 0$ simultaneously and his utility function is given by (A.26).

Note that under Assumption 1, $\mathfrak{U}_i(y_i, \hat{y}_i, \boldsymbol{y}_{-i})$ is strictly concave in y_i for $y_i > \hat{y}_i$ and $y_i < \hat{y}_i$, respectively. Therefore, a sufficient and necessary condition for $\hat{y}_i > 0$ to be a personal equilibrium is

$$\lim_{y_i \searrow \hat{y}_i} \frac{\partial \mathfrak{U}_i(y_i, \hat{y}_i, \boldsymbol{y}_{-i})}{\partial y_i} \leq 0, \text{and} \lim_{y_i \nearrow \hat{y}_i} \frac{\partial \mathfrak{U}_i(y_i, \hat{y}_i, \boldsymbol{y}_{-i})}{\partial y_i} \geq 0,$$

which is equivalent to

$$(1+\eta)\phi_i'(\hat{y}_i) \le v_i\left[1+\eta+\eta(\lambda-1)\frac{\hat{y}_i}{\sum_{j\neq i}^N y_j+\hat{y}_i}\right]\times\frac{\sum_{j\neq i}^N y_j}{\left(\sum_{j\neq i}^N y_j+\hat{y}_i\right)^2}\le(1+\eta\lambda)\phi_i'(\hat{y}_i).$$

For $s>0$, define $\underline{g}_i(s)$ and $\overline{g}_i(s)$ as

$$\underline{g}_i(s) := \begin{cases} 0 & \text{if } \frac{1+\eta}{1+\eta\lambda}v_i \le \phi_i'(0)s, \\ \text{unique positive solution to } \frac{s-y_i}{s^2}\left[1+\eta+\eta(\lambda-1)\frac{y_i}{s}\right]=\frac{1+\eta\lambda}{v_i}\phi_i'(y_i) & \text{otherwise,}\end{cases} \tag{A.27}$$

and

$$\overline{g}_i(s) := \begin{cases} 0 & \text{if } v_i \le \phi_i'(0)s, \\ \text{unique positive solution to } \frac{s-y_i}{s^2}\left[1+\eta+\eta(\lambda-1)\frac{y_i}{s}\right]=\frac{1+\eta}{v_i}\phi_i'(y_i) & \text{otherwise.}\end{cases} \tag{A.28}$$

It can be verified that $\frac{s-y_i}{s^2}\times\left[1+\eta+\eta(\lambda-1)\frac{y_i}{s}\right]$ strictly decreases with y_i for $k\equiv\eta(\lambda-1)\le\frac{1}{3}$. Therefore, both $\underline{g}_i(s)$ and $\overline{g}_i(s)$ are well defined, and it is straightforward to show that $\underline{g}_i(s)\le\overline{g}_i(s)$. Define $g_i^\dagger(s)$ as

$$g_i^\dagger(s) := \begin{cases} \underline{g}_i(s) & \text{if } g_i(s)\le\underline{g}_i(s), \\ g_i(s) & \text{if } \underline{g}_i(s)<g_i(s)<\overline{g}_i(s), \\ \overline{g}_i(s) & \text{if } g_i(s)\ge\overline{g}_i(s),\end{cases} \tag{A.29}$$

where $g_i(s)$ is defined in (A.2) in the proof of Proposition 1. It can be verified that $\frac{\underline{g}_i(s)}{s}$ is strictly decreasing in s for $s<\frac{1}{\phi_i'(0)}\times\frac{1+\eta}{1+\eta\lambda}v_i$ and is equal to zero otherwise. Similarly, $\frac{\overline{g}_i(s)}{s}$ is strictly decreasing in s for $s<\frac{1}{\phi_i'(0)}\times v_i$ and is equal to zero otherwise. Recall that $\frac{g_i(s)}{s}$ is strictly decreasing in s for $s<\frac{1-k}{\phi_i'(0)}\times v_i$ and is equal to zero otherwise. Therefore, $\frac{g_i^\dagger(s)}{s}$ is strictly decreasing in s for $s<\frac{v_i}{\phi_i'(0)}$ and is equal to zero otherwise.

Note that $\widehat{\mathfrak{U}}_i(y_i,\boldsymbol{y}_{-i}):=\mathfrak{U}_i(y_i,y_i,\boldsymbol{y}_{-i})$ is strictly concave in y_i for all $i\in\mathcal{N}$ under Assumption 1 for $k\in[0,\frac{1}{3}]$. Therefore, $\boldsymbol{y}^{**}\equiv(y_1^{**},\ldots,y_N^{**})$ constitutes a PPNE of the modified contest game if and only if $s^{**}:=\sum_{i=1}^N y_i^{**}$ satisfies

$$\sum_{i=1}^N\frac{g_i^\dagger(s^{**})}{s^{**}}=1, \text{and } y_i^{**}=g_i^\dagger\left(s^{**}\right), \forall i\in\mathcal{N}.$$

It remains to show that there exists a unique positive solution to $\sum_{i=1}^N\frac{g_i^\dagger(s)}{s}=1$, which follows from the monotonicity of $\sum_{i=1}^N\frac{g_i^\dagger(s)}{s}$, $\lim_{s\searrow0}\sum_{i=1}^N\frac{g_i^\dagger(s)}{s}=N>1$, and $\sum_{i=1}^N\frac{g_i^\dagger(s)}{s}=0<1$ for $s\ge\frac{v_1}{\phi_i'(0)}$. This concludes the proof. □

Proof of Proposition 13

Proof. With slight abuse of notation, denote the CPNE under η by $\boldsymbol{x}^*(\eta)\equiv\left(x_1^*(\eta),\ldots,x_N^*(\eta)\right)$; and let $y_i^*(\eta):=f_i\left(x_i^*(\eta)\right)$ and $s^*(\eta):=\sum_{i=1}^N y_i^*(\eta)$ for all $i\in\mathcal{N}$. It suffices to verify that the unique CPNE is also a PPNE of the contest game for sufficiently small $\eta>0$, holding fixed $\lambda>1$. It can be verified that the set of active contestants under sufficiently small η coincides with that under $\eta=0$. Without loss of generality, we can assume that the contestants are ordered with respect to $\frac{v_i}{\phi_i'(0)}$, that is,

$$\frac{v_1}{\phi_1'(0)}\ge\cdots\ge\frac{v_N}{\phi_N'(0)}.$$

Then there exists a cutoff $\widehat{\tau}$ such that $x_i^*(0)>0$ for $i\le\widehat{\tau}$ and $x_i^*(0)=0$ otherwise.

First, consider an active contestant $i\in\{1,\ldots,\widehat{\tau}\}$. It suffices to show that $\underline{g}_i(s)<g_i(s)<\overline{g}_i(s)$ from (A.2) and (A.29) as $\eta\searrow0$, which is equivalent to

$$\frac{s^*(\eta)-y_i^*(\eta)}{\left[s^*(\eta)\right]^2}\left[1-\eta(\lambda-1)+2\eta(\lambda-1)\frac{y_i^*(\eta)}{s^*(\eta)}\right]>\frac{1}{1+\eta\lambda}\times\frac{s^*(\eta)-y_i^*(\eta)}{\left[s^*(\eta)\right]^2}\left[1+\eta+\eta(\lambda-1)\frac{y_i^*(\eta)}{s^*(\eta)}\right],$$

and

$$\frac{s^*(\eta)-y_i^*(\eta)}{\left[s^*(\eta)\right]^2}\left[1-\eta(\lambda-1)+2\eta(\lambda-1)\frac{y_i^*(\eta)}{s^*(\eta)}\right]<\frac{1}{1+\eta}\times\frac{s^*(\eta)-y_i^*(\eta)}{\left[s^*(\eta)\right]^2}\left[1+\eta+\eta(\lambda-1)\frac{y_i^*(\eta)}{s^*(\eta)}\right],$$

from (A.27) and (A.28). The first inequality is equivalent to

$$\frac{y_i^*(\eta)}{s^*(\eta)}>\frac{\eta\lambda}{1+2\eta\lambda},$$

which clearly holds for sufficiently small η due to the fact that $\lim_{\eta\searrow 0}\frac{y_i^*(\eta)}{s^*(\eta)}=\frac{y_i^*(0)}{s^*(0)}>0=\lim_{\eta\searrow 0}\frac{\eta\lambda}{1+2\eta\lambda}$. Similarly, the second inequality can be simplified as

$$\frac{y_i^*(\eta)}{s^*(\eta)}<\frac{1+\eta}{1+2\eta},$$

which also holds for sufficiently small η due to the fact that $\lim_{\eta\searrow 0}\frac{y_i^*(\eta)}{s^*(\eta)}=\frac{y_i^*(0)}{s^*(0)}<1=\lim_{\eta\searrow 0}\frac{1+\eta}{1+2\eta}$.

Next, consider an inactive contestant $i\in\mathcal{N}\setminus\{1,\ldots,\widehat{\tau}\}$. Note that $g_i\left(s^*(\eta)\right)=0$ for sufficiently small η; together with (A.2), we have that $v_i\le\phi_i'(0)\times\frac{s^*(\eta)}{1-\eta(\lambda-1)}$ for $i\ge\widehat{\tau}+1$ for sufficiently small η. Consider the two following cases depending on $v_{\widehat{\tau}+1}$ relative to $\phi_{\widehat{\tau}+1}'(0)s^*(0)$.

Case (a): $\boldsymbol{v_{\widehat{\tau}+1}<\phi_{\widehat{\tau}+1}'(0)s^*(0)}$. Then $v_i<\phi_i'(0)s^*(0)$ for $i\ge\widehat{\tau}+1$. Moreover, $\frac{1+\eta}{1+\eta\lambda}v_i\le\phi_i'(0)s^*(\eta)$ for sufficiently small η. Otherwise, suppose, to the contrary, that $\frac{1+\eta}{1+\eta\lambda}v_i>\phi_i'(0)s^*(\eta)$ for sufficiently small η. Then we have

$$v_i=\lim_{\eta\searrow 0}\frac{1+\eta}{1+\eta\lambda}v_i\ge\lim_{\eta\searrow 0}\left[\phi_i'(0)s^*(\eta)\right]=\phi_i'(0)s^*(0),$$

which contradicts $v_i<\phi_i'(0)s^*(0)$. Therefore, $\underline{g}_i\left(s^*(\eta)\right)=0$ for sufficiently small η from (A.27). It follows immediately from (A.29) that $g_i^\dagger\left(s^*(\eta)\right)=0$ for sufficiently small η.

Case (b): $\boldsymbol{v_{\widehat{\tau}+1}=\phi_{\widehat{\tau}+1}'(0)s^*(0)}$. We focus on the case in which $\frac{v_{\widehat{\tau}+1}}{\phi_{\widehat{\tau}+1}'(0)}>\frac{v_{\widehat{\tau}+2}}{\phi_{\widehat{\tau}+2}'(0)}$; the analysis for the case in which $\frac{v_{\widehat{\tau}+1}}{\phi_{\widehat{\tau}+1}'(0)}=\frac{v_{\widehat{\tau}+2}}{\phi_{\widehat{\tau}+2}'(0)}$ is similar. For ease of exposition, let $v_{\widehat{\tau}+2}:=0$ if $\widehat{\tau}+2>N$. Then there exists $\Delta>0$ such that $\frac{v_{\widehat{\tau}+1}-\Delta}{\phi_{\widehat{\tau}+1}'(0)}>\frac{v_{\widehat{\tau}+2}}{\phi_{\widehat{\tau}+2}'(0)}$. Consider the following vector of prize valuations:

$$\boldsymbol{v}_\Delta\equiv(v_1,\ldots,v_{\widehat{\tau}},v_{\widehat{\tau}+1}-\Delta,v_{\widehat{\tau}+2},\ldots,v_N).$$

In words, all contestants except contestant $\widehat{\tau}+1$ have the same prize valuations under $\boldsymbol{v}\equiv(v_1,\ldots,v_N)$ and $\boldsymbol{v}_\Delta$, whereas contestant $\widehat{\tau}+1$'s prize valuation under $\boldsymbol{v}_\Delta$ is less than that under $\boldsymbol{v}\equiv(v_1,\ldots,v_N)$. It is straightforward to verify that the unique CPNE under $\boldsymbol{v}_\Delta$ coincides with that under $\boldsymbol{v}$ for sufficiently small η from (A.2). Similarly, the unique PPNE under $\boldsymbol{v}_\Delta$ coincides with that under $\boldsymbol{v}$ for sufficiently small η from (A.2), (A.27), (A.28), and (A.29). Further, we can conclude from the analysis in case (a) that the unique pure-strategy CPNE under $\boldsymbol{v}_\Delta$ coincides with the unique pure-strategy PPNE under $\boldsymbol{v}_\Delta$ for sufficiently small η. Therefore, the unique pure-strategy CPNE under $\boldsymbol{v}$ also constitutes the unique pure-strategy PPNE under $\boldsymbol{v}$ for sufficiently small η. This concludes the proof. □

Appendix B. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.geb.2022.01.018.

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