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Optimally biased contests with draws☆

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ABSTRACT

We investigate the optimal design of a generalized lottery contest that incorporates the possibility of draws. The designer can impose identity-dependent treatments – i.e., multiplicative biases and additive headstarts – and/or introduce the possibility of draws. For a general objective function, the designer does not allow for draws in the optimal contest.

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1. Introduction

In a contest, players vie for a prize by sinking irreversible effort. A plethora of economic and political interactions exemplify and can be modeled as a contest, ranging from sporting events to college admissions, electoral competitions, rent-seeking activities, and R&D races. An enormous amount of scholarly effort has been devoted to exploring a classical research question: How should a designer bias the competition to boost performance in a contest, especially when contestants are heterogeneous?[1]

One way to boost contest performance is to impose *identity-dependent* preferential treatments on contestants – tailored to their individual characteristics – to vary their relative competitiveness. For example, government can favor small- and medium-sized enterprises in public procurement to support local entrepreneurship (Che and Gale, 2003; Epstein et al., 2011), and colleges can adopt an affirmative action admissions rule to allocate bonus points to minority applicants (Fu, 2006; Franke, 2012; Fryer and Loury, 2005). A vast literature investigates the optimal design of contests when the designer can affect the strategic interactions among contenders by imposing *multiplicative biases* and/or *additive headstarts*; this work includes Franke et al. (2013, 2014, 2018), Kirkegaard (2012), Fu and Wu (2020), Deng et al. (2021), Fu and Wu (2021); and Zhu (2021), among others.

Another way to influence the competition and contestants' strategic choice of effort is to introduce the possibility of draws *independent* of contestants' identity. Contests that permit *draws* (or, interchangeably, ties and gaps) – in which case the prize may be not allocated among contestants – are commonly observed in real-world competitions and have been extensively studied in the literature. Examples include Blavatskyy (2010), Eden (2006), Imhof and Kräkel (2014), Nalebuff and Stiglitz (1983), Lazear and Rosen (1981), Gelder et al. (2019), Cohen and Sela (2007), Jia (2012); and Vesperoni and Yildizparlak (2019), among others. Nti (1997) considers a symmetric multiplayer contest and shows that adding the possibility of a draw lowers competition between contestants and weakens their effort incentive. Deng et al. (2018) extend (Nti, 1997) by introducing player heterogeneity and show that introducing draws in a contest can be optimal for a designer who aims to maximize the expected winner's effort.[2]

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1 See Corchón and Serena (2018), Chowdhury et al. (2023); and Fu and Wu (2019) for recent surveys of theoretical studies of contest design.

2 Relatedly, Eden (2006); Imhof and Kräkel (2014); and Nalebuff and Stiglitz (1983) consider the principal–agent framework and elaborate on the role of draws in saving the agency cost.

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While the optimal contest design for each of the aforementioned instruments has been studied extensively, the *joint design* of the two instruments remains largely underexplored. This paper aims to fill the gap. Specifically, we consider a generalized Tullock contest with concave impact functions, in which players may differ in their prize valuations, impact functions, and effort cost functions. The designer is allowed to impose identity-dependent treatments – i.e., multiplicative biases and additive headstarts – and/or introduce the possibility of draws. Under mild assumptions (Assumptions 1 and 2) on the designer's objective function, we show that the designer will only make use of identity-dependent treatments and draws will not be introduced in the optimal contest (Proposition 1). We also consider a contest objective that takes concern about effort supply, selection efficiency, and closeness of the competition into account and violates one of the assumptions (Assumption 2), and demonstrate that our result continues to hold (Proposition 2). To the best of our knowledge, we are the first to bridge the gap between studies of contest design that allow identity-dependent preferential treatment and the contest literature that permits draws. Our results shed light on the *substitutability* between these two popularly adopted instruments for contest design.

The remainder of the paper is organized as follows: Section 2 describes the model and lays out the optimal contest design problem. Section 3 characterizes the optimal contest and presents the main results. Section 4 concludes.

2. The model

There are $n \geq 2$ risk-neutral contestants $i \in \mathcal{N} := \{1, 2, \ldots, n\}$ competing for a prize. The value of the prize to contestant $i \in \mathcal{N}$ is $v_i > 0$, which is common knowledge. To win the prize, contestants simultaneously submit their effort entry $x_i \geq 0$, which incurs a cost of $c_i(x_i)$. Contestant i's effort cost function $c_i(x_i)$ is twice differentiable, with $c_i(0) = 0$, $c_i'(x_i) > 0$, and $c_i''(x_i) \geq 0$ for all $x_i > 0$.

The winner is selected through a generalized lottery contest. Specifically, fixing an effort profile $\boldsymbol{x} = (x_1, \ldots, x_n)$, contestant $i \in \mathcal{N}$ wins with a probability

$$p_i(\boldsymbol{x}) = \begin{cases} \dfrac{\alpha_i h_i(x_i) + \beta_i}{\sum_{j=1}^n \left[\alpha_j h_j(x_j) + \beta_j\right] + \delta} & \text{if } \sum_{j=1}^n \left[\alpha_j h_j(x_j) + \beta_j\right] + \delta > 0, \\ \dfrac{1}{n} & \text{if } \sum_{j=1}^n \left[\alpha_j h_j(x_j) + \beta_j\right] + \delta = 0, \end{cases} \tag{1}$$

with $\alpha_i \geq 0$, $\beta_i \geq 0$ for all $i \in \mathcal{N}$ and $\delta \geq 0$; the function $h_i(\cdot)$ is typically labeled the impact function of the contest, which is assumed to be twice differentiable, with $h_i(0) \geq 0$, $h_i'(x_i) > 0$, and $h_i''(x_i) \leq 0$ for all $x_i \geq 0$.[3] The set of weights $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} \equiv (\beta_1, \ldots, \beta_n)$ and δ are design variables to be chosen by the designer prior to the competition. The tuple (α_i, β_i) is the *identity-dependent treatment* imposed on each contestant: α_i is a *multiplicative bias* and β_i is an *additive headstart*. The variable $\delta \geq 0$ enables the contest designer to accommodate the possibility of a draw *independent* of contestants' identity: When $\delta > 0$, a draw occurs with a positive probability, in which case the prize is not allocated to contestants; when $\delta = 0$, the prize is distributed to one contestant with certainty.[4]

Given the contest success function (1) and the effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, contestant i's expected payoff is

$$\pi_i(\boldsymbol{x}) = p_i(\boldsymbol{x}) \cdot v_i - c_i(x_i). \tag{2}$$

Contest objectives. Fixing an arbitrary contest rule that consists of $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n \setminus \{(0, \ldots, 0)\}$, $\boldsymbol{\beta} \equiv (\beta_1, \ldots, \beta_n) \in \mathbb{R}_+^n$, and $\delta \in \mathbb{R}_+$, Cornes and Hartley (2005), Jensen (2016); and Szidarovszky and Okuguchi (1997) establish the equilibrium existence and uniqueness of the contest game.[5] The contest designer, anticipating contestants' equilibrium plays, chooses the contest rule $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \delta)$ to maximize an objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$. The contest objective depends on contestants' effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, the profile of winning probabilities $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$, and the profile of prize valuations $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$. We impose the following regularity condition on $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$.

Assumption 1. Fixing $\boldsymbol{p}$ and $\boldsymbol{v}$, $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$ is weakly increasing in x_i for all $i \in \mathcal{N}$.

Assumption 1 requires that contestants' efforts accrue to the benefit of the contest designer. The objective function encompasses a wide array of scenarios. For example, consider the situation in which the designer aims to maximize aggregate effort in the contest—i.e., $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) = \sum_{i=1}^n x_i$—or the expected winner's effort in the contest—i.e., $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) = \sum_{i=1}^n p_i x_i$. The former objective – which only consists of contestants' effort profile $\boldsymbol{x}$ – has been conventionally adopted in the contest literature. The latter – which consists of both the equilibrium effort profile $\boldsymbol{x}$ and the profile of contestants' winning probabilities $\boldsymbol{p}$ – is gaining increasing attention (see, e.g., Baye and Hoppe, 2003; Serena, 2017; Fu and Wu, 2022; Deng et al., 2018; Wasser and Zhang, 2023).

The profile of contestants' prize valuations may also enter into the designer's objective. To see this, consider the following objective function:

$$\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) := \sum_{i=1}^n x_i + \psi \sum_{i=1}^n p_i v_i - \gamma \sum_{i=1}^n \left[p_i - \bar{p}\right]^2, \tag{3}$$

with $\psi \geq 0$ and $\gamma \geq 0$,

where $\bar{p} := \frac{1}{n} \sum_{j=1}^n p_j$. The objective (3) is considered in Deng et al. (2021) and Fu and Wu (2020). In the case of $\psi = \gamma = 0$, the above expression boils down to an objective of total effort maximization. In addition to total effort, the contest designer may be concerned about selection efficiency and/or closeness of the competition. The term $\sum_{i=1}^n p_i v_i$ strictly increases when a contestant of a higher valuation is able to win more often, which takes the concern about selection efficiency with $\psi > 0$ into account.[6] The term $\sum_{i=1}^n [p_i - \bar{p}]^2$ refers to the variance of the winning probabilities of the contestants. With $\gamma > 0$, the designer prefers a less predictable outcome. Concern about the closeness often emerges in sports contests: Spectators often not only appreciate contestants' effort but also demand more suspense about the eventual winners.[7] Simple algebra would verify that contest objective (3) satisfies Assumption 1.

3. Analysis

The traditional optimization approach does not apply in our context, because it requires a closed-form solution to equilibrium effort profile $\boldsymbol{x}$ for every possible contest rule $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \delta)$ (see,

[3] Contest success function (1) is axiomatized in Blavatskyy (2010) and Vesperoni and Yildizparlak (2019). Jia (2012) and Jia et al. (2013) provide a stochastic foundation for (1).

[4] The parameter $\delta \geq 0$ is interpreted as a discount rate in the context of a patent race in Nti (1997).

[5] Cornes and Hartley (2005) and Szidarovszky and Okuguchi (1997) establish the existence and uniqueness of the equilibrium in the above contest game with $\delta = 0$ (see also Fu and Wu, 2020 and Stein, 2002). Jensen (2016) further proves equilibrium existence and uniqueness for the case of $\delta > 0$.

[6] For contest design for selection efficiency, see Hvide and Kristiansen (2003), Meyer (1991), Fang and Noe (2022); and Ryvkin and Ortmann (2008).

[7] For economics studies of suspense in competition, see Ely et al. (2015), Chan et al. (2009), Fort and Quirk (1995); and Szymanski (2003).

e.g., Franke et al., 2013, 2014, 2018). However, a multi-player asymmetric contest in general cannot be solved in closed form. To proceed, we adopt the approach recently proposed by Fu and Wu (2020) and Deng et al. (2021) to bypass this technical difficulty, which allows us to characterize the optimum without explicitly solving for the equilibrium. The key to the approach is to (i) first reformulate contestants' first-order conditions and establish a correspondence between contestants' effort profile $\boldsymbol{x}$ and profile of winning probabilities $\boldsymbol{p}$ in the unique equilibrium; then (ii) treat $\boldsymbol{p}$ as the design variable and express the objective $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$ as a function of $\boldsymbol{p}$ only.

The first-order condition $\partial\pi_i/\partial x_i = 0$ for an active contestant i – who exerts a strictly positive amount of effort – is

$$p_i(1-p_i)v_i = c_i'(x_i)\frac{\alpha_i h_i(x_i)+\beta_i}{\alpha_i h_i'(x_i)}, \text{ for } x_i > 0, \tag{4}$$

and that for an inactive contestant i – who exerts zero effort in equilibrium – gives

$$p_i(1-p_i)v_i \le c_i'(x_i)\frac{\alpha_i h_i(x_i)+\beta_i}{\alpha_i h_i'(x_i)}, \text{ for } x_i = 0. \tag{5}$$

The following lemma ensues.

Lemma 1 (*Suboptimality of Additive Headstarts*). *Suppose that Assumption 1 is satisfied. Then the optimum can be achieved by choosing $(\boldsymbol{\alpha}, \delta)$ only and setting $\boldsymbol{\beta} = (0, \dots, 0)$.*

Proof. The proof closely follows that of Theorem 2 in Fu and Wu (2020) and is omitted for brevity. □

By Lemma 1, it is without loss of generality to consider the optimal design of the contest rule by abstracting away additive headstarts – i.e., by assuming $\boldsymbol{\beta} = (0, \dots, 0)$ – and focusing on the design of $\boldsymbol{\alpha}$ and δ. Substituting $\beta_i = 0$ into the first-order condition (4) yields

$$p_i(1-p_i)v_i = c_i'(x_i) \times \frac{h_i(x_i)}{h_i'(x_i)}. \tag{6}$$

It is noteworthy that if a contestant stands zero chance of winning the contest, it must be the case that the contestant exerts zero effort. Therefore, the correspondence (6) between a contestant's winning probability p_i and his equilibrium effort x_i also holds for an inactive contestant who exerts zero effort.

For notational convenience, denote the inverse function of $c_i'(x)h_i(x)/h_i'(x)$ as $g_i(\cdot)$. It can be verified that $g_i(\cdot)$ is a strictly increasing function.[8] The correspondence (6) can then be written as

$$x_i = g_i\left(p_i(1-p_i)v_i\right). \tag{7}$$

By (7), fixing $\boldsymbol{v}$, we can calculate the unique equilibrium effort profile – which we denote by $\boldsymbol{x}(\boldsymbol{p}, \boldsymbol{v})$ – once the profile of equilibrium winning probabilities $\boldsymbol{p}$ is known.

The next lemma states that any profile of equilibrium winning probabilities $\boldsymbol{p} \equiv (p_1, \dots, p_n) \ge (0, \dots, 0)$, with $0 < \sum_{i=1}^n p_i \le 1$, can be implemented by some $(\boldsymbol{\alpha}, \delta)$, which we refer to as the contest rule hereafter for ease of exposition.

Lemma 2 (*Implementing Equilibrium Winning Probabilities Through Contest Design*). *Fix an arbitrary profile of equilibrium winning probabilities $\boldsymbol{p} \equiv (p_1, \dots, p_n) \in \{(p_1, \dots, p_n) \mid p_i \ge 0$ for all $i \in \mathcal{N}$ and $0 < \sum_{i=1}^n p_i \le 1\}$.*

(i) If $p_j = 1$ for some $j \in \mathcal{N}$, then $\boldsymbol{p} \equiv (p_1, \dots, p_n)$ can be induced by the set of biases $\boldsymbol{\alpha}(\boldsymbol{p}) \equiv (\alpha_1(\boldsymbol{p}), \dots, \alpha_n(\boldsymbol{p}))$

$$\alpha_i(\boldsymbol{p}) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{8}$$

and $\delta(\boldsymbol{p}) = 0$.

(ii) If $p_j \neq 1$ for all $j \in \mathcal{N}$, then $\boldsymbol{p} \equiv (p_1, \dots, p_n)$ can be induced by the set of biases $\boldsymbol{\alpha}(\boldsymbol{p}) \equiv (\alpha_1(\boldsymbol{p}), \dots, \alpha_n(\boldsymbol{p}))$

$$\alpha_i(\boldsymbol{p}) = \begin{cases} \dfrac{p_i}{h_i\left[g_i\left(p_i(1-p_i)v_i\right)\right]} & \text{if } p_i > 0, \\ 0 & \text{if } p_i = 0, \end{cases} \tag{9}$$

and $\delta(\boldsymbol{p}) = 1 - \sum_{i=1}^n p_i$.

Proof. The proof is analogous to that of Theorem 3 in Fu and Wu (2020) and is omitted for brevity. □

By Lemma 2, the contest designer can construct a contest rule $(\boldsymbol{\alpha}, \delta)$ to induce any profile of equilibrium winning probabilities $\boldsymbol{p}$. The lemma, together with (7), implies that (i) the design problem can be reformulated into one in which the designer chooses $\boldsymbol{p}$ to maximize the objective function $\Lambda(\boldsymbol{x}(\boldsymbol{p}, \boldsymbol{v}), \boldsymbol{p}, \boldsymbol{v})$; and (ii) upon obtaining the maximizer to $\Lambda(\boldsymbol{x}(\boldsymbol{p}, \boldsymbol{v}), \boldsymbol{p}, \boldsymbol{v})$, the optimal contest rule can be pinned down by invoking Lemma 2. To obtain more mileage, we impose the following condition on $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$.

Assumption 2. Fixing $\boldsymbol{x}$ and $\boldsymbol{v}$, $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$ is weakly increasing in p_i for all $i \in \mathcal{N}$.

It can be verified that the popularly studied objective of total effort maximization—i.e., $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) = \sum_{i=1}^n x_i$—and expected winner's effort maximization—i.e., $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) = \sum_{i=1}^n p_i x_i$—satisfy Assumption 2. Denote the optimal contest rule that maximizes $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$ by $(\boldsymbol{\alpha}^*, \delta^*)$. The following result can be obtained.

Proposition 1 (*Suboptimality of Introducing Draws in Contests*). *Suppose that Assumptions 1 and 2 are satisfied. Then the optimum can be achieved by setting $\delta^* = 0$.*

Proof. Suppose, to the contrary, that $\delta^* > 0$ in the optimal contest and no contest rule $(\boldsymbol{\alpha}, \delta)$, with $\delta = 0$, can achieve the optimum. Denote the equilibrium effort profile and the associated profile of equilibrium winning probabilities under $(\boldsymbol{\alpha}^*, \delta^*)$ by $\boldsymbol{x}^* \equiv (x_1^*, \dots, x_n^*)$ and $\boldsymbol{p}^* \equiv (p_1^*, \dots, p_n^*)$, respectively. Further, assume without loss of generality that contestants are ordered such that $p_1^* \le p_2^* \le \dots \le p_n^*$. It follows immediately from the postulated $\delta^* > 0$ that $\sum_{i=1}^n p_i^* < 1$.

Next, we construct an alternative equilibrium winning probability profile $\boldsymbol{p}' = (p_1', \dots, p_n')$, with $\sum_{i=1}^n p_i' = 1$ and the associated effort profile $\boldsymbol{x}' = (x_1', \dots, x_n')$ derived from (7) such that $x_i' \ge x_i^*$ and $p_i' \ge p_i^*$ for all $i \in \mathcal{N}$. We consider the following two cases depending on p_n^* relative to 1/2.

Case I: $\boldsymbol{p_n^* \ge 1/2}$. It follows immediately that $\sum_{i=1}^{n-1} p_i^* < 1/2$. Let $p_j' = p_j^*$ for $j \in \mathcal{N}\backslash\{n-1\}$ and $p_{n-1}' = p_{n-1}^* + \left(1 - \sum_{j=1}^n p_j^*\right)$. It is straightforward to verify that $p_i^* = p_i'$ for all $i \in \mathcal{N}\backslash\{n-1\}$, $p_{n-1}^* < p_{n-1}' \le 1/2$, and $\sum_{i=1}^n p_i' = 1$. Moreover, by condition (7), we have that $x_i' = x_i^*$ for all $i \in \mathcal{N} \setminus \{n-1\}$ and

$$x_{n-1}' = g_i\left[p_{n-1}'(1-p_{n-1}')v_{n-1}\right] > g_i\left[p_{n-1}^*(1-p_{n-1}^*)v_{n-1}\right] = x_{n-1}^*,$$

where the strict inequality follows from $p_{n-1}^* < p_{n-1}' \le 1/2$ and the monotonicity of $g_i(\cdot)$.

Case II: $\boldsymbol{p_n^* < 1/2}$. Let $p_i' = p_i^*$ for all $i \in \mathcal{N} \setminus \{n-1, n\}$. Further, if $\sum_{j=1}^{n-1} p_j^* \ge 1/2$, we set $(p_{n-1}', p_n') = \left(p_{n-1}^*, 1 - \sum_{j=1}^{n-1} p_j^*\right)$. If

[8] To see this, note that the weak convexity of the effort cost function $c_i(x)$ implies that $c_i'(x)$ is weakly increasing in x. Further, it can be verified from the strict monotonicity and weak concavity of the impact function $h_i(x)$ that $h_i(x)/h_i'(x)$ is strictly increasing in x. Therefore, $c_i'(x)h_i(x)/h_i'(x)$ strictly increases with x, from which we can conclude the strict monotonicity of $g_i(\cdot)$.

otherwise $\sum_{j=1}^{n-1} p_j^* < 1/2$, we set $(p'_{n-1}, p'_n) = \left(p^*_{n-1} + (1/2 - \sum_{j=1}^{n-1} p_j^*), 1/2\right)$. Simple algebra would verify that $p_i^* \le p'_i \le 1/2$ for all $i \in \mathcal{N} \setminus \{n\}$, $p_n^* < p'_n \le 1/2$, and $\sum_{j=1}^{n} p'_j = 1$. Together with condition (7), we can obtain that $x'_i \ge x_i^*$ for all $i \in \mathcal{N} \setminus \{n\}$ and $x'_n > x_n^*$.

In summary, we can construct an alternative equilibrium winning probability profile $\boldsymbol{p}' \equiv (p'_1, \ldots, p'_n) \ge (p_1^*, \ldots, p_n^*) \equiv \boldsymbol{p}^*$, with $\sum_{i=1}^{n} p'_i = 1$, such that $\boldsymbol{x}' \equiv (x'_1, \ldots, x'_n) \equiv (x_1^*, \ldots, x_n^*) \equiv \boldsymbol{x}^*$. From $\sum_{i=1}^{n} p'_i = 1$ and Lemma 2, the contest rule to induce $(\boldsymbol{x}', \boldsymbol{p}')$ – which we denote by $(\boldsymbol{\alpha}', \delta')$ – must satisfy $\delta' = 0$. That is, the possibility of draws is not introduced in the contest. Moreover, it follows from Assumptions 1 and 2 that $\Lambda(\boldsymbol{x}', \boldsymbol{p}', \boldsymbol{v}) \ge \Lambda(\boldsymbol{x}^*, \boldsymbol{p}^*, \boldsymbol{v})$. This concludes the proof. □

By Proposition 1, the designer has no incentive to introduce draws in a contest if she can impose identity-dependent preferential treatment on contestants. This result stands in sharp contrast to that of Deng et al. (2018), who show that introducing the possibility of draws can effectively incentivize strong contestants and increase the expected winner's effort if their values of winning are sufficiently dispersed, given that she is not able to vary the competitive balance of the playing field through imposition of identity-dependent treatments.[9] Intuitively, introducing draws in a contest renders every contestant less likely to win, ceteris paribus. This would incentivize stronger contestants to step up their efforts, while discouraging weaker contestants from bidding. Given that stronger contestants have a higher equilibrium winning probability than weaker contestants *absent* identity-dependent preferential treatments, the designer may benefit from the imposition of draws when the resulting increase in equilibrium effort from stronger contestants is substantial.

Notably, Proposition 1 states that the imposition of draws in a contest would lose its appeal when identity-dependent preferential treatments are allowed for contest design. The economic logic underlying this result can be interpreted from the perspective of contestants' equilibrium winning probabilities. Intuitively, the benefit of introducing the possibility of draws is plagued by its "waste" of contestants' winning probabilities: Contestants' equilibrium winning probabilities do not sum up to one and the prize is not distributed with a positive probability. Recall the correspondence (7). A contestant's effort decision ultimately hinges on the expectation regarding how likely he is to win. Once the contest designer can impose multiplicative biases on contestants, she can incentivize contenders by manipulating the competitive balance of the playing field. In particular, she can appropriately assign the "wasted" equilibrium winning probabilities triggered by the imposition of draws to spur incentive provision.

Two remarks are in order. First, recall that the contest objective of the expected winner's effort maximization considered by Deng et al. (2018) satisfies Assumptions 1 and 2. Our Proposition 1 thus accentuates the *substitutability* between the two types of design instruments that manipulate the contest success function (1)—i.e., between imposing identity-dependent multiplicative biases on heterogeneous contestants and introducing the possibility of draws in a contest.

Second, Assumption 2 is crucial in establishing Proposition 1. However, the objective function as specified in (3) does not satisfy the assumption.[10] It remains unclear a prior whether the suboptimality of introducing draws established in Proposition 1 remains intact if the assumption is violated. Our next result addresses this concern.

[9] Deng et al. (2018) also show that introducing draws in a contest always leads to a decrease in the equilibrium total effort.

[10] To see this, note that when $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$ takes the form of (3), we have that $\partial\Lambda/\partial p_i = \psi v_i - 2\gamma(p_i - \bar{p})$, the sign of which turns negative when $p_i > \bar{p}$ for sufficiently small ψ and sufficiently large γ.

Proposition 2 (*Suboptimality of Introducing Draws for a Contest Objective That Violates Assumption 2*)**.** *Suppose that the contest designer's objective function takes the form of (3). Then $\delta^* = 0$ in the optimal contest.*

Proof. For notational convenience, let $\mathcal{D}(\boldsymbol{p}) := \sum_{j=1}^{n} \left(p_j - \bar{p}\right)^2$. Contest objective (3) can then be written as

$$\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) \equiv \sum_{i=1}^{n} x_i + \psi \sum_{i=1}^{n} p_i v_i - \gamma \mathcal{D}(\boldsymbol{p}), \text{ with } \psi \ge 0 \text{ and } \gamma \ge 0.$$

Fix an arbitrary profile of equilibrium winning probabilities $\boldsymbol{p} \equiv (p_1, \ldots, p_n) \ge (0, \ldots, 0)$, with $\bar{p} \equiv (\sum_{j=1}^{n} p_j)/n < 1/n$ and the associated equilibrium effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$ that satisfies (7). Note that $\bar{p} < 1/n$ indicates that $\delta > 0$ for the contest rule that induces $\boldsymbol{p}$. Assume without loss of generality that contestants are ordered such that $p_1 \le \cdots \le p_n$. Further, let $m := \max\{j : p_j \le \bar{p}, j \in \mathcal{N}\}$; $r := \max\{j : p_j \le \frac{1}{2}\left(\bar{p} + \frac{1}{n}\right), j \in \mathcal{N}\}$; and $s := \max\{j : p_j \le 1/n, j \in \mathcal{N}\}$. It follows immediately that $m \le r \le s$. For expositional convenience, we focus on the case in which the inequalities hold strictly – i.e., $m < r < s$ – in the subsequent analysis. The analysis for cases in which some of the equalities hold is similar.

The proof consists of two steps. In the first, we construct an alternative profile of equilibrium winning probabilities $\boldsymbol{p}' := (p'_1, \ldots, p'_s, p'_{s+1}, \ldots, p'_n)$, with $\sum_{j=1}^{n} p'_j = 1$, such that $p'_i \ge p_i$ for all $i \le s$ and $p'_i = p_i$ for all $i > s$. In the second step, we show that $\Lambda(\boldsymbol{x}', \boldsymbol{p}', \boldsymbol{v}) > \Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$, where $\boldsymbol{x}' := (x'_1, \ldots, x'_n)$ is the associated equilibrium effort profile that satisfies (7).

Step I. We construct an alternative profile of equilibrium winning probabilities $\bar{p}' := (p'_1, \ldots, p'_n)$, with $\sum_{j=1}^{n} p'_j = 1$, such that

$$p_i \le p'_i \le 1/n \text{ and } \left(p'_i - 1/n\right)^2 \le (p_i - \bar{p})^2, \text{ for } i \le s; \tag{10}$$

and

$$p'_i = p_i, \text{ for } i > s. \tag{11}$$

Fix $i \le s$. It can be verified that condition (10) is equivalent to

$$\frac{1}{n} - |p_i - \bar{p}| \le p'_i \le \frac{1}{n}, \text{ and } p'_i \ge p_i. \tag{12}$$

Simple algebra would verify that condition (12) boils down to (i) $p'_i \in \left[\frac{1}{n} + p_i - \bar{p}, \frac{1}{n}\right]$ for $1 \le i \le m$; (ii) $p'_i \in \left[\frac{1}{n} + \bar{p} - p_i, \frac{1}{n}\right]$ for $m+1 \le i \le r$; and (iii) $p'_i \in \left[p_i, \frac{1}{n}\right]$ for $r+1 \le i \le s$.

Define $\mathcal{T}(q_1, \ldots, q_s) := \sum_{j=1}^{s} q_j$, where

$$(q_1, \ldots, q_s) \in \mathcal{Q} := \prod_{j=1}^{m} \left[\frac{1}{n} + p_j - \bar{p}, \frac{1}{n}\right] \times \prod_{j=m+1}^{r} \left[\frac{1}{n} + \bar{p} - p_j, \frac{1}{n}\right] \times \prod_{j=r+1}^{s} \left[p_j, \frac{1}{n}\right].$$

It follows immediately that

$$\min_{(q_1,\ldots,q_s)\in\mathcal{Q}} \mathcal{T}(q_1, \ldots, q_s) = \sum_{j=1}^{m} \left(\frac{1}{n} + p_i - \bar{p}\right) + \sum_{j=m+1}^{r} \left(\frac{1}{n} + \bar{p} - p_j\right) + \sum_{j=r+1}^{s} p_j,$$

and

$$\max_{(q_1,\ldots,q_s)\in\mathcal{Q}} \mathcal{T}(q_1, \ldots, q_s) = \frac{s}{n}.$$

Note that

$$\begin{aligned}
&\left(1-\sum_{j=s+1}^{n} p_j\right) - \min_{(q_1,\ldots,q_s)\in\mathcal{Q}} \mathcal{T}(q_1,\ldots,q_s)\\
=&1-\sum_{j=s+1}^{n} p_j - \sum_{j=1}^{m}\left(\frac{1}{n}+p_j-\bar{p}\right) - \sum_{j=m+1}^{r}\left(\frac{1}{n}+\bar{p}-p_j\right)\\
&-\sum_{j=r+1}^{s} p_j\\
=&1-\sum_{j=r+1}^{n} p_j - \sum_{j=1}^{m}\left(\frac{1}{n}+p_j-\bar{p}\right) - \sum_{j=m+1}^{r}\left(\frac{1}{n}+\bar{p}-p_j\right)\\
=&1-\frac{r}{n}-\left(\sum_{j=1}^{m} p_j + \sum_{j=r+1}^{n} p_j - \sum_{j=m+1}^{r} p_j\right) + \left(\sum_{j=1}^{m}\bar{p} - \sum_{j=m+1}^{r}\bar{p}\right)\\
=&1-\frac{r}{n}-\left(\sum_{j=1}^{m} p_j + \sum_{j=r+1}^{n} p_j + \sum_{j=m+1}^{r} p_j - 2\sum_{j=m+1}^{r} p_j\right)\\
&+\left(\sum_{j=1}^{m}\bar{p} + \sum_{j=m+1}^{r}\bar{p} - 2\sum_{j=m+1}^{r}\bar{p}\right)\\
=&1-\frac{r}{n}-\left(n\bar{p}-2\sum_{j=m+1}^{r} p_j\right) + \left(r\bar{p}-2\sum_{j=m+1}^{r}\bar{p}\right)\\
=&(n-r)\left(\frac{1}{n}-\bar{p}\right)+2\sum_{j=m+1}^{r}(p_j-\bar{p}) \geq 0,
\end{aligned} \tag{13}$$

where the inequality follows from $\bar{p} < 1/n$ and $p_i > \bar{p}$ for all $i \in \{m+1,\ldots,r\}$.

Further, we have that

$$\begin{aligned}
&\left(1-\sum_{j=s+1}^{n} p_j\right) - \max_{(q_1,\ldots,q_s)\in\mathcal{Q}} \mathcal{T}(q_1,\ldots,q_s)\\
&=\left(1-\sum_{j=s+1}^{n} p_j\right) - \frac{s}{n} \leq 0,
\end{aligned} \tag{14}$$

where the inequality from $p_i > 1/n$ for all $i \in \{s+1,\ldots,n\}$.

Combining (13) and (14) yields that

$$\min_{(q_1,\ldots,q_s)\in\mathcal{Q}} \mathcal{T}(q_1,\ldots,q_s) \leq 1-\sum_{j=s+1}^{n} p_j \leq \max_{(q_1,\ldots,q_s)\in\mathcal{Q}} \mathcal{T}(q_1,\ldots,q_s); \tag{15}$$

together with the intermediate value theorem, we can conclude that there exists a vector $(q_1',\ldots,q_s') \in \mathcal{Q}$ such that $\mathcal{T}(q_1',\ldots,q_s') = 1-\sum_{j=s+1}^{n} p_j$.

Let $\boldsymbol{p}' \equiv (p_1',\ldots,p_s',p_{s+1}',\ldots,p_n') = (q_1',\ldots,q_s',p_{s+1},\ldots,p_n)$. It is evident that the constructed $\boldsymbol{p}'$ satisfies (10), (11), and $\sum_{j=1}^{n} p_j' = \mathcal{T}(q_1',\ldots,q_s') + \sum_{j=s+1}^{n} p_j = 1$.

Step II. From (10) and (11), we have that $p_i' \geq p_i$ for all $i \in \mathcal{N}$, which in turn implies that

$$\sum_{i=1}^{n} p_i' v_i \geq \sum_{i=1}^{n} p_i v_i. \tag{16}$$

Define $\bar{p}' := \frac{1}{n}\sum_{j=1}^{n} p_j'$. Recall $\sum_{j=1}^{n} p_j' = 1$ by our construction in Step I and $\bar{p} < 1/n$. As a result, $\bar{p}' = 1/n > \bar{p}$. For $i \leq s$, it follows immediately from (10) that $\left(p_i'-\bar{p}'\right)^2 = \left(p_i'-1/n\right)^2 \leq (p_i-\bar{p})^2$. Further, for $i > s$, we have $p_i' = p_i$ and $p_i > 1/n = \bar{p}' > \bar{p}$, which implies that $\left(p_i'-\bar{p}'\right)^2 < (p_i-\bar{p})^2$. Taken together, we can obtain that

$$\mathcal{D}(\boldsymbol{p}') = \sum_{i=1}^{n}\left(p_i'-\bar{p}'\right)^2 \leq \sum_{i=1}^{n}(p_i-\bar{p})^2 = \mathcal{D}(\boldsymbol{p}). \tag{17}$$

From $\sum_{j=1}^{n} p_j' = 1 > \sum_{j=1}^{n} p_j$ and $p_i' = p_i$ for $i \geq s+1$, there exists some contestant $k \leq s$ such that $p_k' > p_k$. Together with the facts that $g_i(\cdot)$ is a strictly increasing function and $p_i' \leq 1/n \leq 1/2$ for all $i \leq s$, we have that

$$g_i\left(p_i'(1-p_i')v_i\right) \geq g_i\left(p_i(1-p_i)v_i\right), \text{ for } i \in \{1,\ldots,s\}\setminus\{k\} \tag{18}$$

and

$$g_k\left(p_k'(1-p_k')v_k\right) > g_k\left(p_k(1-p_k)v_k\right). \tag{19}$$

Therefore, we have that

$$\begin{aligned}
\sum_{i=1}^{n} x_i' &= \sum_{i=1}^{n} g_i\left(p_i'(1-p_i')v_i\right)\\
&= \sum_{i=1}^{s} g_i\left(p_i'(1-p_i')v_i\right) + \sum_{i=s+1}^{n} g_i\left(p_i'(1-p_i')v_i\right)\\
&> \sum_{i=1}^{s} g_i\left(p_i(1-p_i)v_i\right) + \sum_{i=s+1}^{n} g_i\left(p_i(1-p_i)v_i\right) = \sum_{i=1}^{n} x_i,
\end{aligned} \tag{20}$$

where the strict inequality follows from (18) and (19).

Combining (16), (17), and (20) yields that

$$\begin{aligned}
\Lambda(\boldsymbol{x}',\boldsymbol{p}',\boldsymbol{v}) &\equiv \sum_{i=1}^{n} x_i' + \psi\sum_{i=1}^{n} p_i' v_i - \gamma\mathcal{D}(\boldsymbol{p}')\\
&> \sum_{i=1}^{n} x_i + \psi\sum_{i=1}^{n} p_i v_i - \gamma\mathcal{D}(\boldsymbol{p}) \equiv \Lambda(\boldsymbol{x},\boldsymbol{p},\boldsymbol{v}).
\end{aligned}$$

The above analysis, together with Lemma 2, implies that for an arbitrary contest rule $(\boldsymbol{\alpha},\delta)$, with $\delta > 0$, we can always construct an alternative contest rule $(\boldsymbol{\alpha}',\delta')$, with $\delta' = 0$, that generates a higher payoff to the contest designer. This concludes the proof. □

By Proposition 2, introducing draws in a contest always reduces contest performance, given that the designer's objective function is in the form of (3). Despite the complexity caused by the nonmonotonicity of the contest objective with respect to a contestant's equilibrium probability p_i, we can again adjust the contest rule to "redistribute" contestants' equilibrium winning probabilities and make use of the "wasted" equilibrium winning probability caused by the possibility of draws – as we do in the proof of Proposition 1 – to improve contest performance.

4. Concluding remarks

In this paper, we study the optimal design of a generalized lottery contest that incorporates the possibility of draws. We show that the optimal contest does not allow for draws for a general objective function that encompasses a wide array of scenarios.

Data availability

No data was used for the research described in the article.

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